Asset Selling Under Debt Obligations

Hyun-soo Ahn[*], Derek D. Wang[**], and Owen Q. Wu[***]

[*]Ross School of Business, University of Michigan, Ann Arbor, Michigan, U.S.

[**]1. College of Business Administration, Capital University of Economics and Business, Beijing, China,
2. Desautels Faculty of Management, McGill University, Montreal, Quebec, Canada

[***]Kelley School of Business, Indiana University, Bloomington, Indiana, U.S.

May 7, 2019

We extend the classical asset-selling problem to include debt repayment obligation, selling capacity constraint, and Markov price evolution. Specifically, we consider the problem of selling a divisible asset which is acquired through debt financing. The amount of asset that can be sold per period may be limited by physical constraints. The seller uses part of the sales revenue to repay the debt. If unable to pay off the debt, the seller must go bankrupt and liquidate the remaining asset. Our analysis reveals that in the presence of debt, the optimal asset-selling policy must take into account two opposing forces: an incentive to sell part of the asset early to secure debt payment and an incentive to delay selling the asset to capture revenue potential under limited liability. We analyze how these two forces, originating from debt financing, will distort the seller's optimal policy.

1. Introduction

Financing asset acquisition and selling the asset is a common practice in many industries. In the agricultural industry, farm loans are often used to finance farming operations, while crops are sold to generate revenue, part of which repays the loans. For example, in the Midwest, farmers invest billions of dollars every year in the corn crop. Corn can be dried and stored for over a year, allowing farmers to choose when and how much to sell their crop. Farm bankruptcies are common due to fluctuations in crop prices (Stam and Dixon 2004). In the energy and mining industries, acquisition of mineral rights, exploration and construction of infrastructure constitute the bulk of the setup investment that needs to be financed before any revenue from selling the resources can be realized. For example, a shale gas producer leases land from land owners at a cost that can be as high as \$15,000 per acre, and drilling a well needs 40-80 acres. Drilling, hydraulic fracturing, and well completion cost \$3-7 million per well. Production of gas lasts for 10-20 years, and the producer can control the rate of production to some extent in order to sell more gas at favorable market prices. Interestingly, even though the price of natural gas plunged in 2008, instead of cutting back their production and waiting for the price to recover, producers kept extracting natural gas at a high rate. Among the reasons for producing more in a dire market is that some firms are forced to produce under financial pressure of paying off their debts.

Motivated by these industry practices, we aim to explore how debt obligations affect asset-selling decisions in this paper. We consider a discrete-time problem of selling a divisible asset over a finite horizon. Prior to the first period, the asset acquisition is partially or fully financed by debt, and from period 1 through N, the seller faces a stochastically evolving price process and decides in each period how much of the asset to sell at the ongoing price. The seller uses the sales revenue to repay the debt and must go bankrupt if unable to meet the debt obligation. We analyze the impact of the debt level (the amount that the seller owes) and the debt payment schedule (e.g., timing of debt maturity, pay all at once or in installments) on the seller's optimal selling strategies.

We also consider selling capacity constraint in our model. Capacity constraint imposes a limit on the amount of asset that can be sold per period, which is commonly observed in practice, e.g., selling natural gas from a well or minerals from a mine is limited by the speed of extraction; selling farm crops to the market is limited by the labor and transportation capacity. We analyze how the capacity constraint and debt obligation jointly affect the optimal asset-selling strategies.

1.1 Related Literature

The asset-selling problem, in its basic form as in Karlin (1962), considers selling a single non-divisible asset when n prices independently drawn from a known distribution are presented to the seller sequentially. Upon being offered a price, the seller must decide whether to accept the offer and sell the asset or reject it and wait for the next price, with the goal of maximizing the expected revenue. This problem has many variants (reviewed shortly), and the family of asset-selling problems has been well studied in the context of the stochastic search problem since Stigler (1961). A related search problem is the dowry problem or secretary problem, in which candidates are presented sequentially to a decision-maker, who can rank candidates without tie and must choose or reject each candidate based on the relative ranks observed so far, with the goal of maximizing the probability of choosing the best one.[1] The dowry problem and its variants are examined by Gilbert and Mosteller (1966), Freeman (1983), Chun et al. (2002), among others.

The asset-selling problem is first analyzed by Karlin (1962), who proves that the optimal policy

[1] In the dowry problem, the payoff is essentially 1 if the best is chosen and 0 otherwise (i.e., nothing but the best), whereas in the asset-selling problem, the payoff is the selling price.

is to sell the asset when the offered price exceeds a time-dependent reservation price, which is the maximum expected price attainable in the remaining periods. Gilbert and Mosteller (1966) compare reservation prices under various settings and illustrate that the reservation prices are higher if the right-hand tail of the price distribution is larger. Karlin (1962) also considers an infinite-horizon setting with a discount factor and finds that the reservation price is a constant. Lippman and McCall (1976) consider an infinite-horizon job search problem with a fixed cost per search and find that the optimal policy can be characterized by a reservation wage; see also Telser (1973) for a similar problem of a buyer searching for the lowest price.

Several authors have considered the problem of selling multiple identical assets (no more than one in any period) with the objective of maximizing the total expected revenue. Karlin (1962) and Gilbert and Mosteller (1966) find that the optimal policy is characterized by reservation prices that depend on the number of remaining assets for sale. Selling multiple assets is actually a special case of the sequential assignment problem, in which a decision-maker must assign n known quantities q_i, $i = 1, \ldots, n$ to sequentially revealed random variables X_j, $j = 1, \ldots, n$ (independently drawn from a known distribution) to maximize the total expected payoff, given that pairing q_i with X_j yields a payoff $q_i X_j$. Derman et al. (1972) prove that optimal assignment for X_1 is characterized by n non-overlapping intervals such that if X_1 falls in the ith interval, it is optimal to assign the ith smallest quantity to X_1. Albright (1974) further generalizes the problem to allow random arrival times of X_j. In §4, the asset-selling problem with a selling capacity constraint is related to the problem of selling multiple assets.

For a divisible asset, if the revenue is linear in the amount sold, selling the asset as a whole is still optimal. When the payoff function is concave, however, dividing the asset for sale may be desirable. Derman et al. (1975) analyze a sequential investment problem, which is generalized by Prastacos (1983). In this problem, an investor with a certain amount of capital decides how much to invest in each sequentially revealed opportunity. The quality of each opportunity is independently drawn from a known distribution. Investing q (irreversible) in an opportunity of quality X yields a return $R(q, X)$. Prastacos (1983) considers the special case of $R(q, X) = qX$, which is equivalent to the basic asset-selling problem; he also examines the case when $R(q, X)$ is concave in q and derives the optimal investment strategy. We consider the problem of selling a divisible asset with debt obligations and provide the structure of the optimal policy.

A common assumption in most asset selling models is that the sequentially revealed prices are independently and identically distributed, with only a few exceptions. Karlin (1962) and Lippman and McCall (1976) allow the price to follow a semi-Markov process and show that the optimal policy can be characterized by reservation prices that depend on the state of the underlying Markov process. Pye (1971) models the price evolution as a random walk and considers a different objective: to minimize regret.

In this paper, we model price evolution as a Markov process. We assume that the stochastic properties of the price process are known to the seller. This assumption is also made in the papers reviewed above, but the asset-selling problem with unknown price distribution has also received considerable attention. Heuristic methods are developed by Telser (1973), while Bayesian updating methods are adopted by Rothschild (1974), Albright (1977), and Rosenfield et al. (1983).

There is a continuing interest in the asset-selling problem. Ee (2009) extends the asset-selling problem to allow random termination of the selling process as well as the options of skipping search in a period and terminating search by taking a quitting offer. Palley and Kremer (2014) consider a search problem where the decision-maker knows the distribution of the candidate values but only observes the relative rankings of the candidates until the search stops. The asset-selling problem and its variants have broad applications, including the labor market (Rogerson, Shimer, and Wright 2005), kidney allocation (Su and Zenios 2005), land development (Batabyal and Yoo 2005), and online commerce (Gallien 2006).

A seller with market power can post a selling price and customers whose valuation of the asset exceeds the selling price will buy. Arnold and Lippman (2001) consider the problem of pricing one unit of asset in face of Poisson demand with known distribution of valuation. They also extend the model to selling multiple units over time at posted prices, which is essentially a revenue management problem. Phillips (2005) provides an extensive review of the literature on pricing and revenue management. Below, we review a few papers that introduce financial constraints or targets to the classic revenue management problem studied by Gallego and van Ryzin (1994). Levin et al. (2008) consider selling multiple units over a finite horizon and analyze the efficient frontier of expected revenue and the probability of meeting a revenue target. Besbes and Maglaras (2012) introduce revenue and sales milestone constraints into the revenue management problem and find that the optimal pricing policy dynamically tracks the most stringent future milestone. Besbes et al. (2018) analyze a discrete-time version of the revenue management problem under debt obligations. This paper complements the above research by studying the problem of selling assets at the market price (as opposed to the posted price) under debt obligations.

The effects of debt on operations have been examined in various contexts. Xu and Birge (2004) study a newsvendor problem with debt financing and demonstrate the value of integrating production and financing decisions. Buzacott and Zhang (2004) analyze the interactions between a firm's financing and operation decisions when the maximum amount of the loan is based on the firm's assets. Kouvelis and Zhao (2012) compare bank financing with supplier financing in a newsvendor setting and study the optimal trade credit contracts. Yang et al. (2015) examine how the possibility of bankruptcy impacts product market competition and various parties in the supply chain. Chod (2017) analyzes how debt distorts a retailer's inventory decision and ways to mitigate such distortion. Iancu et al. (2017) finds that firms shielded by limited liability may use operating flexibility at the expense of their creditors, resulting in higher borrowing costs. Besbes et al. (2018) analyze the revenue management problem when the seller has limited liability for a debt repayment at the end of the horizon. They find that the debt induces the seller to price consistently higher than the revenue-maximizing policy, and this distortion increases over time, leading to a downward spiral in the expected revenue. Our research not only considers the effect of limited liability but also captures the cost of dissolution, as described in §1.2 below.

1.2 Our Contributions

This paper extends the classical asset-selling problem to include debt obligations. We analyze two distinct effects of debt obligations on optimal asset-selling policies and the interactions between these effects. The first effect is commonly known as the limited liability effect (Myers 1977, Chod 2017, Besbes et al. 2018, among others). In our context, if the seller is unable to pay off the debt, a straight bankruptcy procedure allows the seller to liquidate the remaining asset and the debt is then discharged. Thus, in the adverse price scenarios, bankruptcy protects the seller from carrying the debt obligations indefinitely. As limited liability curbs the downside loss, the seller tends to delay selling the asset to capture upside revenue potential. The second effect of debt stems from the costs associated with bankruptcy. Bankruptcy incurs direct administrative costs and indirect costs related to the value loss when assets are liquidated (Ang et al. 1982, Bris et al. 2006, Kouvelis and Zhao 2011). Thus, the presence of bankruptcy cost incentivizes the seller to secure capital early to pay off the debt. As a result, the seller deviates from the revenue maximizing strategy and sells (part of) the asset early. To the best of our knowledge, this paper is the first to analyze how limited liability and bankruptcy cost jointly affect the optimal asset selling strategy. Whether the seller delays or expedites selling the asset depends on the relative strength of these two effects, which vary across different debt agreements. When the two effects are equally strong, it is possible that the optimal asset-selling policy under debt coincides with the revenue-maximizing policy.

We also study how the capacity constraint interacts with the two effects of debt obligations and find that the capacity constraint weakens both effects. That is, the magnitude of delayed selling (driven by limited liability) or expedited selling (driven by bankruptcy cost) decreases as the seller's capacity constraint tightens. Furthermore, we find that the presence of capacity constraint may reduce the bankruptcy risk, especially when the capacity constraint is moderate and the debt level is not too high. This result echoes the negative impact of operating flexibility found by Iancu et al.

(2017).

This paper establishes the condition under which selling a divisible asset with capacity constraint is equivalent to the problem of selling multiple non-divisible units. This equivalence allows us to compare the optimal policies for selling assets at the market price versus selling at the posted price. In contrast to Besbes et al. (2018) who find that price distortion increases over time, we show that the distortion on the reservation price may decrease over time, due to the strong downward pressure on the reservation price toward the end of the horizon.

Finally, we derive most of the results under the assumption that the market price evolves according to a Markov process, which is more general than the independent and identical price distributions assumed in most of the existing literature.

2. Asset Selling Model and Debt Financing

We consider a seller (firm) selling a divisible asset over a T-period horizon $\mathcal{T} \stackrel{\text{def}}{=} \{1, 2, \ldots, T\}$. Before the beginning of the horizon (labeled as period 0), the seller makes a one-time investment to acquire the asset. We assume there is no opportunity to acquire additional assets after period 0. Divisibility of the asset is not a critical assumption but facilitates analysis. The qualitative results in this paper continue to hold if the seller has a large number of non-divisible units for sale.

2.1 Asset Selling without Debt Constraints

As a benchmark, we first formulate the problem of selling a divisible asset without debt obligations, i.e., the seller has enough initial capital (through self-financing or equity) to acquire the asset in period 0, which is sold over periods 1 to T. The value of any unsold asset at the end of period T diminishes to zero.

Let $x_t \in [0, 1]$ denote the amount of asset available for sale at the beginning of period t. Without loss of generality, the initial size of the asset is normalized to $x_1 = 1$. Let $P_t \geq 0$ be the random selling price in period t, and let p_t denote its realization. Upon observing p_t at the beginning of period t, the seller decides the selling quantity in period t, denoted as q_t. The maximum amount of asset that can be sold per period is $\ell \in (0, 1]$, i.e., $q_t \in [0, \ell]$. If $\ell = 1$, the seller can sell the entire asset in one period, which is the case we consider first in §3. The case of $\ell < 1$ is studied in §4.

We model the price process $\{P_t : t \in \mathcal{T}\}$ as a discrete-time continuous-state Markov process. Let $F_t(\cdot \mid p_{t-1})$ be the cumulative distribution function of P_t conditioning on the realized price p_{t-1}. We assume $\mathsf{E}[P_t \mid p_{t-1}] < \infty$, for all p_{t-1} and all $t \in \mathcal{T}$.

Let $U_t(x_t, p_t)$ be the maximum discounted expected revenue-to-go from period t onward when the amount of asset for sale at the beginning of period t is x_t and the realized price is p_t. Let $\rho \in (0, 1)$ be the seller's discount factor. Then, $U_t(x_t, p_t)$ can be determined by the following dynamic program:

$$U_t(x_t, p_t) = \max_{0 \le q_t \le \min(x_t, \ell)} p_t q_t + \rho \mathsf{E}_t U_{t+1}(x_t - q_t, P_{t+1}), \quad t = 1, ..., T, \tag{1}$$
$$U_{T+1}(.,.) = 0,$$

where E_t is the expectation conditioning on the observed price p_t.

2.2 Asset Selling Under Debt Constraints

If the firm cannot raise enough capital for investment, it can finance with a debt in period 0 and repay the debt using the revenue from selling the asset. Similar to Besbes et al. (2018), we assume a debt contract is already in place and analyze the selling decision under the debt. Let m ($1 \le m \le T$) be the debt maturity period. The debt payment schedule is denoted as $\mathbf{d} = \{d_t : t = 1, \ldots, m\}$, where $d_t \ge 0$ is the installment to be paid at the end of period t. If unable to pay the debt, the seller files for bankruptcy under Chapter 7 and the remaining asset is liquidated.

Let w_t denote the seller's working capital at the beginning of period t. The seller bankrupts in period t if $w_t + p_t q_t < d_t$, i.e., the sum of current working capital and sales revenue cannot cover the debt payment. We do not allow debt renegotiation in the model. For the ease of exposition, we set the initial working capital $w_1 = 0$. This assumption does not change the results qualitatively but brings notational convenience. Indeed, if $w_1 > 0$, it can be shown that the problem can be transformed to an equivalent asset-selling problem with $w_1 = 0$ and reduced debt levels.

Let $V_t(x_t, w_t, p_t; \mathbf{d})$, $t \le m$, denote the equity value of the firm (i.e., the value accruing to the firm's shareholders) at the beginning of period t, with inventory x_t, working capital w_t, and realized price p_t. The value function must incorporate the firm's working capital and revenue-to-go as well as the debt obligations and bankruptcy risks, which are not reflected in standard asset-selling problems.

To derive the value function, for $t \le m$, let $\underline{q_t} \overset{\text{def}}{=} (d_t - w_t)^+ / p_t$ be the minimum sales quantity needed to pay debt d_t in period t. Suppose the seller survives through periods 1 to $m-1$. In period m (debt maturity), the seller can survive if and only if $\underline{q_m} \le \min(x_m, \ell)$. If this condition holds, the seller chooses $q_m \in \left[\underline{q_m}, \min(x_m, \ell)\right]$ to pay off the debt and continues to sell the remaining asset (if any) from period $m+1$ onward without debt obligation. Thus, for $t = m+1, \ldots, T$, the optimal selling policy can be determined by (1). If $\underline{q_m} > \min(x_m, \ell)$, the seller is unable to pay off the debt and goes bankrupt. Thus, the value function in period m is defined as:

$$V_m(x_m, w_m, p_m; \mathbf{d}) =$$
$$\begin{cases} \max\limits_{\underline{q_m} \le q_m \le \min(x_m, \ell)} p_m q_m + w_m - d_m + \rho \mathsf{E}_m U_{m+1}(x_m - q_m, P_{m+1}), & \text{if } p_m \min(x_m, \ell) + w_m \ge d_m, \\ 0, & \text{if } p_m \min(x_m, \ell) + w_m < d_m. \end{cases} \tag{2}$$

Note that the survival condition $p_m \min(x_m, \ell) + w_m \geq d_m$ in (2) is equivalent to $\underline{q}_m \leq \min(x_m, \ell)$.

In (2), we assume that the revenue from the liquidation sale cannot cover the debt (i.e., the indirect bankruptcy cost in terms of the loss in asset value is high), but the seller is shielded by limited liability. Thus, the equity value diminishes to zero upon bankruptcy, which is a common assumption in the literature (e.g., Xu and Birge 2004, Boyabatli and Toktay 2011, and Chod and Zhou 2013). This assumption automatically holds when $m = T$ (recall that unsold asset at the end of period T has no value). When $m < T$, we will extend the model in §5 to consider the liquidation process, which allows the seller to collect residual liquidation revenue after the debt is paid off. The analysis for the extended model confirms that key qualitative results continue to hold.

In each period $t < m$, if $\underline{q}_t \leq \min(x_t, \ell)$, the seller sells $q_t \in \left[\underline{q}_t, \min(x_t, \ell)\right]$ and uses the revenue to pay the debt d_t. The resulting working capital $w_t + p_t q_t - d_t$ grows at its internal rate of return $1/\rho$. If $\underline{q}_t > \min(x_t, \ell)$, the seller fails to pay d_t and goes bankrupt. Thus, the dynamic program for $t < m$ can be written as

$$V_t(x_t, w_t, p_t; \mathbf{d}) = \begin{cases} \max\limits_{\underline{q}_t \leq q_t \leq \min(x_t, \ell)} \rho \mathsf{E}_t V_{t+1}\big(x_t - q_t,\ \rho^{-1}(w_t + p_t q_t - d_t),\ P_{t+1}; \mathbf{d}\big), & \text{if } p_t \min(x_t, \ell) + w_t \geq d_t, \\ 0, & \text{if } p_t \min(x_t, \ell) + w_t < d_t. \end{cases} \tag{3}$$

Note from (3) that having enough working capital in period t (e.g., $w_t \geq d_t - p_t \min(x_t, \ell)$) only guarantees that bankruptcy does not occur in period t.

Intuitively, if working capital w_t is high enough to cover all of the remaining debt payments, then the equity value should be linear in w_t. This intuition is confirmed in part (i) of the following lemma. Proofs of all lemmas and propositions are included in the Online Appendix. Throughout this paper, monotonicity and convexity are used in their weak sense.

Lemma 1 *In period $t \leq m$,*

(i) if $w_t \geq \sum_{i=t}^{m} \rho^{i-t} d_i$, then $V_t(x_t, w_t, p_t; \mathbf{d}) = U_t(x_t, p_t) + w_t - \sum_{i=t}^{m} \rho^{i-t} d_i$,

(ii) $V_t(0, w_t, p_t; \mathbf{d}) = \left(w_t - \sum_{i=t}^{m} \rho^{i-t} d_i\right)^+$, and

(iii) $V_t(x_t, w_t, p_t; \mathbf{d})$ is increasing in w_t and x_t, but neither convex nor concave in general.

Lemma 1 suggests that a working capital level at or above $\sum_{i=t}^{m} \rho^{i-t} d_i$ removes bankruptcy risk from period t onward and, therefore, the seller shall follow the revenue-maximizing policy determined in (1) from period t onward (Besbes et al. 2018 obtain a similar result in a posted-price setting). Part (ii) provides a simple formula for the firm's equity value when the asset is sold out before period $t \leq m$. In general, the value function is neither convex nor concave; see examples in §3.3 and §4.2.

3. Asset Selling without Capacity Constraint

In this section, we consider the asset-selling problem without capacity constraint, i.e., $\ell = 1$. We first analyze the debt-free asset-selling problem formulated in (1) and then analyze how the debt obligation and bankruptcy risk alter the optimal selling policy. To this end, we consider the case of a single debt payment at maturity $m = T$ and show the limited liability effect. Then, we compare that with the case of a single debt payment at $m < T$ and demonstrate the effect of bankruptcy cost. We then extend our study to general debt payment schedules.

3.1 Debt-Free Asset-Selling Policy

In the classical problem of selling a non-divisible asset under independently distributed prices (Karlin 1962), the optimal selling policy obeys the one-time stopping rule characterized by a reservation price (which can depend on t): The asset is sold whenever the price exceeds the reservation price. We generalize this result for the case of a divisible asset and Markov price evolution.

For each period t, we define R_t^o as the maximum expected revenue from selling all of the asset *after* period t, discounted back to period t:

$$R_t^o = \rho\, \mathsf{E}\left[U_{t+1}(1, P_{t+1}) \mid P_t\right]. \tag{4}$$

We sometimes write $R_t^o(P_t)$ to emphasize its dependence on P_t.

Proposition 1 *(i) When there is no debt obligation (i.e., $\mathbf{d} = \mathbf{0}$), the optimal asset-selling policy is to sell the entire asset in period $\tau^o \stackrel{\text{def}}{=} \inf\{t : P_t \geq R_t^o,\, t \in \mathcal{T}\}$.*
(ii) $R_T^o = 0$ and $R_t^o = \rho \mathsf{E}_t \max\{P_{t+1}, R_{t+1}^o\}$ for $t = 1, \ldots, T-1$. The expected best selling price discounted to the present, $\mathsf{E}_0 \rho^t R_t^o$, decreases in t.
(iii) (Karlin 1962) If the prices P_t, $t \in \mathcal{T}$, are independently and identically distributed (i.i.d.), then R_t^o is deterministic and decreases in t.

Proposition 1(i) shows that it is optimal to sell the asset all at once. Thus, R_t^o can be interpreted as the expected discounted best selling price after period t, and the optimal selling time is the first time when P_t exceeds R_t^o. The iterative relation $R_t^o = \rho \mathsf{E}_t \max\{P_{t+1}, R_{t+1}^o\}$ in part (ii) implies that the expected discounted best selling price is no lower than the maximum of the discounted expected prices: $R_t^o \geq \max\{\rho \mathsf{E}_t P_{t+1}, \ldots, \rho^{T-t} \mathsf{E}_t p_T\}$, due to the Jensen's inequality. Part (iii) shows that our results are consistent with the classical asset-selling problem under i.i.d. prices.

In the optimal stopping rule in Proposition 1(i), although R_t^o appears to play the role of a reservation price, R_t^o in general varies with P_t due to the Markov price process, and thus R_t^o is not a predetermined reservation price. We refer to R_t^o as the *critical price* for the debt-free case.

Prior research (see §1.1) typically assumes independent price distributions, in which case R^o_t has a deterministic value, which is the reservation price.

A natural question is under what conditions there exists a reservation price (that can be predetermined) above which the asset should be sold. Proposition 2 answers this question.

Proposition 2 *Suppose for every $t = 1, \ldots, T-1$, (i) P_{t+1} stochastically increases in p_t, i.e., $F_{t+1}(x \,|\, p_t)$ decreases in p_t, $\forall x \geq 0$, and (ii) $\mathsf{E}[P_{t+1} \mid p_t]$ increases in p_t at a rate no greater than 1. Then, for every t, $R^o_t(p_t)$ increases in p_t at a rate no greater than $\rho \in (0,1)$, and there exists a unique $\widehat{p}_t$ such that $p_t \geq R^o_t(p_t)$ if and only if $p_t \geq \widehat{p}_t$. The optimal policy is to sell the entire asset in period $\tau^o = \inf\{t : P_t \geq \widehat{p}_t,\, t \in \mathcal{T}\}$.*

Thus far, we have shown that the structure of the optimal policy for selling a divisible asset under Markov price evolution is similar to that in the classical models. Next, we explore how the introduction of debt obligation affects the optimal policy, focusing on the changes in the critical prices. Comparing critical prices is analytically tractable and yields the same insights as comparing the implicit reservation prices, because a higher critical price corresponds to a higher reservation price.

3.2 Single Debt Payment at $m = T$

When the debt requires a single payment at the end of the horizon, i.e., $m = T$, bankruptcy can occur only in period T. We will show that the asset is still sold all at once, but the debt obligation affects the timing of the sale, which can be attributed to the limited liability effect.

Lemma 2 *Suppose the debt obligation requires a single payment at the end of period T, i.e., $d_t = 0$ for $t = 1, \ldots, T-1$ and $d_T > 0$. Then, for $t \in \mathcal{T}$,*
(i) $V_t(x_t, w_t, p_t; \mathbf{d})$ is jointly convex in (x_t, w_t);
(ii) $V_t(x_t, w_t, p_t; \mathbf{d}) \geq U_t(x_t, p_t) + w_t - \rho^{T-t} d_T$, with equality holding if $w_t \geq \rho^{T-t} d_T$.

The convexity in Lemma 2(i) is essential for the one-time stopping rule to continue to hold, which will be formalized in Proposition 3. The convexity of the value function implies that inventory has increasing marginal value. When the entire inventory can be sold in one period, a higher inventory level reduces bankruptcy probability and, thus, enhances the marginal value of inventory. (In §4.1 when selling capacity exists, inventory exhibits diminishing marginal value.)

In Lemma 2 (ii), if $w_t \geq \rho^{T-t} d_T$, i.e., the working capital is sufficient to cover the debt payment, the seller can simply follow the revenue-maximization strategy in Proposition 1 from period t onward and obtain an expected value of $U_t(x_t, p_t) + w_t - \rho^{T-t} d_T$, which is consistent with Lemma 1(i). However, if $w_t < \rho^{T-t} d_T$ (bankruptcy risk exists), part (ii) shows that the revenue maximization strategy is not necessarily optimal because the limited liability protects the firm's equity value from dropping below zero, resulting in $V_t(x_t, w_t, p_t; \mathbf{d}) > U_t(x_t, p_t) + w_t - \rho^{T-t} d_T$.

The next proposition characterizes the optimal policy and compares it with the debt-free case.

Proposition 3 *Suppose the debt obligation requires a single payment at the end of period T, i.e., $d_t = 0$ for $t = 1, \ldots, T-1$ and $d_T > 0$. Then,*
(i) There exists a series of critical prices $\{R_t : t \in \mathcal{T}\}$ with $R_T = d_T$ and $R_t = \rho \mathsf{E}_t \max\{P_{t+1}, R_{t+1}\}$, such that the seller should sell the entire asset in period $\tau = \inf\{t : P_t \geq R_t, t \in \mathcal{T}\}$, which is an optimal stopping time. Upon selling, the revenue will ensure debt payment in period T. If $P_t < R_t$ for all $t \in \mathcal{T}$ (i.e., $\tau = \infty$), the seller goes bankrupt at the end of period T.
(ii) When the debt d_T increases, the critical price R_t increases almost surely, the stopping time τ increases almost surely, and the probability of bankruptcy increases. In particular, $R_t \geq R_t^o$ and $\tau \geq \tau^o$.
(iii) The expected critical price discounted to the present, $\mathsf{E}_0 \rho^t R_t$, decreases in t. If prices P_t, $t \in \mathcal{T}$, are i.i.d., then R_t is the deterministic reservation price and there exists a debt level $\overline{d}$, such that R_t is constant over time for $d_T = \overline{d}$, R_t decreases in t for $d_T < \overline{d}$, and R_t increases in t for $d_T > \overline{d}$.

Proposition 3(i) confirms that the optimal policy under a single debt payment at $m = T$ still follows the one-time stopping rule. Furthermore, the relation $R_{t-1} = \rho \mathsf{E} \max\{P_t, R_t\}$ holds at all debt levels, including the zero-debt case in Proposition 1.

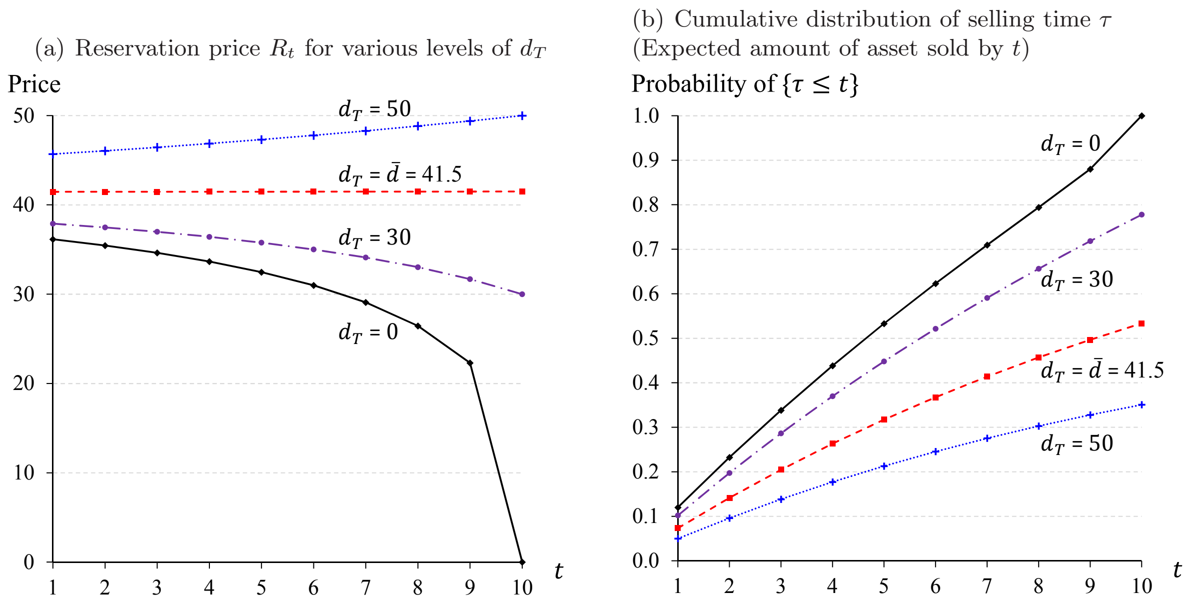
Importantly, Proposition 3(ii) reveals that debt obligation delays the optimal selling time compared to the debt-free case, and a higher debt results in a longer delay in selling the asset. Intuitively, the downside risk of delaying the sale is reduced due to the firm's limited liability for the debt, whereas waiting keeps the seller open to the upside potential. Consider a situation in period t when the seller decides whether or not to sell the asset. A debt-free seller can earn p_t by selling now or earn an expected value of $\rho \mathsf{E}_t U_{t+1}(1, P_{t+1})$ by waiting. With debt d_T, selling now brings a net value of $p_t - \rho^{T-t} d_T$, while waiting brings an expected value of $\rho \mathsf{E}_t V_{t+1}(1, 0, P_{t+1}; \mathbf{d})$, higher than $\rho \mathsf{E}_t U_{t+1}(1, P_{t+1}) - \rho^{T-t} d_T$ due to Lemma 2(ii), which means the seller has more incentive to wait than in a debt-free situation. Besbes et al. (2018) also find that limited liability delays sales in a posted-price setting.

Proposition 3(iii) highlights how debt obligation changes the reservation price for the case of i.i.d. selling prices. In contrast to the no-debt case in Proposition 1(iii), the reservation price can increase or decrease in t. At a low debt level, the reservation price decreases in t, because with low bankruptcy risk, the main driver for asset-selling decision is still to maximize sales revenue. At a high debt level, the main driver for selling the asset is to cover the debt payment, that is, the selling price in period t must be at least $\rho^{T-t} d_T$ (which increases in t) to cover the debt, which causes the reservation price R_t to increase over time, as confirmed in Proposition 3(iii). Note that, if there is no discounting as in Besbes et al. (2018), the reservation price would always decrease over time.

Figure 1 illustrates the results in Proposition 3. Panel (a) shows that the reservation price in any period increases in the debt level d_T (proved in Proposition 3(ii)) and that there exists a threshold debt level $\overline{d}$, above (below) which R_t increases (decreases) in t, as verified in Proposition 3(iii). Furthermore, observe that the difference between the reservation price under debt and the debt-free reservation price increases over time. This result echoes the increasing distortion of posted prices in Besbes et al. (2018), but contrasting results will be discussed in the next sections.

Figure 1: Optimal asset-selling policy under single debt payment: $m = T = 10$

P_t's are i.i.d. lognormal and $\log(P_t) \sim \mathcal{N}(3, 0.5)$, discount factor $\rho = 0.98$



Panel (b) shows the cumulative distribution of τ, $\Pr\{\tau \leq t\}$, which is also the expected amount of asset sold by period t, $\Pr\{\tau \leq t\} \cdot 1 + \Pr\{\tau > t\} \cdot 0$, because the entire asset is sold at τ. As debt d_T increases, $\Pr\{\tau \leq t\}$ decreases at every t, implying that the seller delays selling the asset.

3.3 Single Debt Payment at $m < T$

In this case, if the firm is unable to meet the debt payment in period m, it must go bankrupt and the firm's value diminishes to zero (see (2)). The optimal decision for the case of $m < T$ is driven by both the limited liability analyzed in §3.2 and the seller's desire to avoid costly bankruptcy by selling a portion of the asset early to pay off the debt.

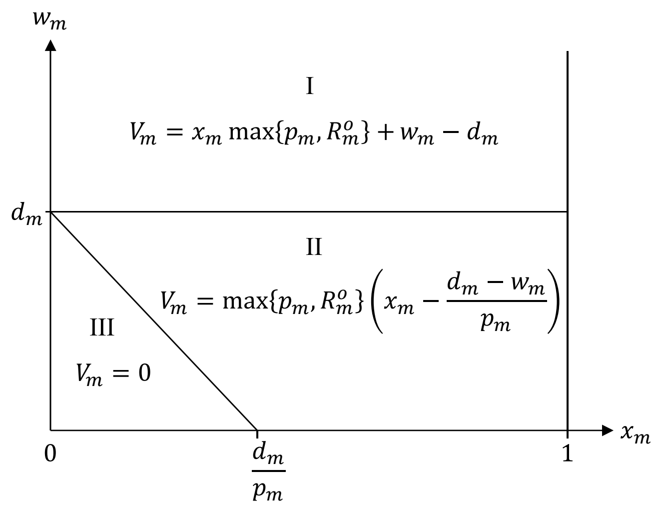
In this section, we analyze the intricate trade-off between the benefit of limited liability and bankruptcy cost. In preparation, we first prove the properties of the value function.

Lemma 3 *Suppose the debt obligation requires a single payment d_m at the end of period $m < T$. Then, for $t \leq m$, $V_t(x_t, w_t, p_t; \mathbf{d})$ is convex in (x_t, w_t, d_m) in region $w_t \leq \rho^{m-t} d_m$ and is linear in (x_t, w_t, d_m) in region $w_t \geq \rho^{m-t} d_m$. $V_t(x_t, w_t, p_t; \mathbf{d})$ is continuous in (x_t, w_t, d_m).*

Figure 2 shows the value function in period m, with explicit expressions derived in the proof of Lemma 3. Note that the value function is convex in (x_m, w_m) across regions II and III but is concave across regions I and II when $p_m < R_m^o$ (in Region II the value function increases in w_m at slope $\frac{R_m^o}{p_m} > 1$, whereas in Region I it increases in w_m at unit slope). Backward induction through (3) preserves the convexity of the value function across regions II and III ($w_t < \rho^{m-t} d_m$) and linearity in region I ($w_t > \rho^{m-t} d_m$), but the value function is not convex in general.

Figure 2: Value function $V_m(x_m, w_m, p_m; \mathbf{d})$, $\mathbf{d} = (0, \ldots, 0, d_m)$

Region I: working capital w_m can cover the debt d_m.
Region II: w_m is insufficient to cover the debt, and the seller must sell $\underline{q}_m = \frac{d_m - w_m}{p_m}$ to pay off the debt.
Region III: $\underline{q}_m > x_m$ (or $p_m x_m + w_m < d_m$), the firm is unable to pay off the debt and goes bankrupt.



Lemma 3 establishes the convexity of $V_t(x_t, w_t, p_t; \mathbf{d})$ for two distinct regions $w_t \leq \rho^{m-t} d_m$ and $w_t \geq \rho^{m-t} d_m$, leading to a different optimal policy structure formalized in Proposition 4. (Lemma 3 includes d_m in the convexity, which is needed for studying the effect of debt.)

Before formally presenting the structure of optimal policy, it is useful to understand the effects of bankruptcy on the firm's equity value. Upon bankruptcy, the seller loses the future revenue from selling the remaining asset. This indirect bankruptcy cost decreases the expected equity value. On the other hand, limited liability protects the equity value from dropping below zero, which enhances the expected equity value. When $m < T$, both effects exist and the composite effect on the firm's value can be represented by Δ_t, defined as

$$\Delta_t \stackrel{\text{def}}{=} V_t(1, 0, P_t; \mathbf{d}) - \big(U_t(1, P_t) - \rho^{m-t} d_m\big), \qquad t = 1, \ldots, m+1, \tag{5}$$

where we set $V_{m+1}(1,0,P_{m+1};\mathbf{d}) \stackrel{\text{def}}{=} 0$. We define it up to period $m+1$ because the selling decisions up to period m are the focal point of analysis.

Recall Lemma 2(ii) suggests that $V_t(1,0,p_t;\mathbf{d}) \geq U_t(1,p_t) - \rho^{T-t}d_T$ if $m = T$, because a feasible policy is to capture the maximum expected revenue $U_t(1,p_t)$. However, with debt maturity $m < T$, the inequality $V_t(1,0,p_t;\mathbf{d}) \geq U_t(1,p_t) - \rho^{m-t}d_m$ may not hold because the seller may not be able to capture the maximum expected revenue $U_t(1,p_t)$ due to possibility of bankruptcy. Therefore, Δ_t in (5) is exactly the value of limited liability net the value loss due to bankruptcy cost. When $\Delta_t > 0$, the limited liability effect is stronger than the bankruptcy cost effect; when $\Delta_t < 0$, the opposite is true. The joint effect drives the seller to adopt a new selling strategy described next.

Proposition 4 *Suppose the debt obligation requires a single payment d_m at the end of period $m < T$. (i) There exist two series of critical prices $\{(R_t^{(1)}, R_t^{(2)}) : 1 \leq t \leq m\}$ with $R_t^{(1)} \leq R_t^{(2)}$, $1 \leq t \leq m$, such that the seller should make the first sale in period $\tau = \inf\{t : P_t \geq R_t^{(1)}, 1 \leq t \leq m\}$, which is an optimal stopping time, and*

- *If $\tau \leq m$ and $p_\tau \geq R_\tau^{(2)}$, then it is optimal to sell the entire asset in period τ;*
- *If $\tau \leq m$ and $p_\tau < R_\tau^{(2)}$, then it is optimal to sell $q_\tau = \rho^{m-\tau}d_m/p_\tau$ in period τ and sell the remaining asset in period $\tau' = \inf\{t : P_t \geq R_t^o, \tau < t \leq T\}$, where R_t^o is the critical price for the no-debt case, as defined in (4);*
- *If $p_t < R_t^{(1)}$, do not sell. If $p_t < R_t^{(1)}$ for all $t \leq m$ (i.e., $\tau = \infty$), then the seller bankrupts at the end of period m.*

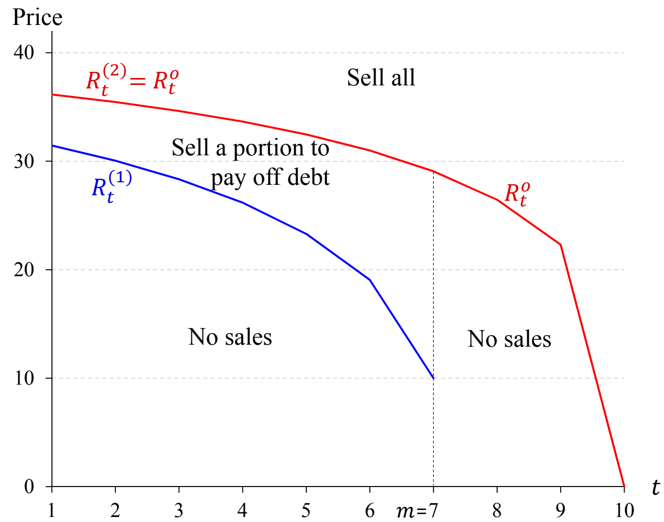
(ii) In period $t \leq m$, if $\mathsf{E}_t\Delta_{t+1} < 0$, then $R_t^{(1)} < R_t^{(2)} = R_t^o$. If $\mathsf{E}_t\Delta_{t+1} \geq 0$, then $R_t^{(1)} = R_t^{(2)} \geq R_t^o$. In particular, if $\mathsf{E}_t\Delta_{t+1} = 0$, then $R_t^{(1)} = R_t^{(2)} = R_t^o$.

Proposition 4(i) prescribes that the optimal policy for the first sale is a control band policy (illustrated in Figure 3): if the price is sufficiently high (above $R_t^{(2)}$), sell the entire asset; if the price is very low (below $R_t^{(1)}$), sell nothing; if the price falls in between the two critical prices, it is optimal to sell part of the asset to secure debt payment, i.e., sell $q_\tau = \rho^{m-\tau}d_m/p_\tau$ to earn $\rho^{m-\tau}d_m$ which will cover the debt d_m. Thus, in this middle belt, a higher price p_τ leads to (counter-intuitively) a lower selling quantity q_τ, which is driven by the strategy of generating revenue to cover the debt payment exactly.

Part (ii) of the proposition further illuminates that, depending on which of the two forces is stronger, the critical prices and the width of the control band are significantly different. When the bankruptcy cost dominates the limited liability benefit ($\mathsf{E}_t\Delta_{t+1} < 0$), the seller has a strong incentive to avoid costly bankruptcy by selling part of the asset early to secure the debt payment. This partial sale, if it occurs, will be *earlier* than the sale in the no-debt case because the critical

Figure 3: Optimal asset-selling policy under single debt payment: $T = 10$, $m = 7$

P_t's are i.i.d. lognormal and $\log(P_t) \sim \mathcal{N}(3, 0.5)$, $\rho = 0.98$, $d_m = 10$



price $R_t^{(1)}$ is below R_t^o, which is the critical price for the no-debt case defined in (4). After this partial sale removes the bankruptcy risk, the seller will sell the remaining asset using the same policy as in the no-debt case. Therefore, no sale will occur later than the selling time in the debt-free case. This strategy is in sharp contrast with the delayed selling decision when the debt matures in period T (see Proposition 3).

On the other hand, when the limited liability effect is stronger ($\mathsf{E}_t\Delta_{t+1} > 0$), the seller does not worry about the bankruptcy cost as much. Consequently, the partial selling region disappears. But, as in the $m = T$ case, bankruptcy shields the seller from the downside risk, which delays the sales compared to the no-debt case ($R_t^{(1)} = R_t^{(2)} \geq R_t^o$).

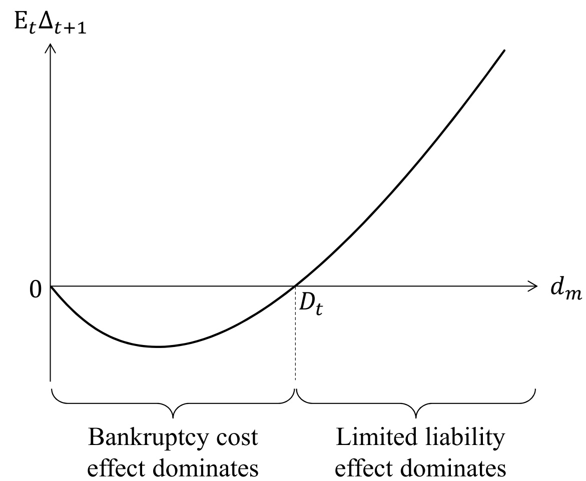
Proposition 4 proves that the sales should be expedited (delayed) if $\mathsf{E}_t\Delta_{t+1} < 0\,(> 0)$, i.e., the bankruptcy cost effect is stronger (weaker) than the limited liability effect. A natural question is what drives the relative magnitude of these two effects. One may expect that $\mathsf{E}_t\Delta_{t+1}$ is monotone in the debt level d_m, since a higher debt leads to more benefit from limited liability. It turns out that $\mathsf{E}_t\Delta_{t+1}$ is not monotone and relates to d_m in a manner depicted in Figure 4.

When $d_m = 0$, Lemma 1(i) implies that $V_{t+1}(1, 0, P_{t+1}; \mathbf{0}) = U_{t+1}(1, P_{t+1})$ for $t < m$, and thus $\mathsf{E}_t\Delta_{t+1} = 0$. Figure 4 illustrates that $\mathsf{E}_t\Delta_{t+1}$ decreases in d_m first, which is proved next.

Lemma 4 *For all $t < m$, we have* $\lim_{d_m \to 0^+} \dfrac{\partial \mathsf{E}_t\Delta_{t+1}}{\partial d_m} \leq 0$. *Furthermore, the inequality is strict if* $\mathsf{Pr}\{P_t < R_t^o\} > 0$ *for all* $t \leq m$.

In Lemma 4, the condition $\mathsf{Pr}\{P_t < R_t^o\} > 0$ for all $t < m$ is satisfied by most price processes. In particular, the condition is satisfied if the support of the price distribution includes low price levels.

Figure 4: Relationship between $\mathsf{E}_t\Delta_{t+1}$ and debt level



Thus, $\mathsf{E}_t\Delta_{t+1}$ strictly decreases in d_m for small d_m, as shown in Figure 4. Therefore, for small debt levels, $\mathsf{E}_t\Delta_{t+1} < 0$, implying that the effect of bankruptcy cost is more pronounced than the limited liability effect.

Figure 4 shows that $\mathsf{E}_t\Delta_{t+1}$ is convex in d_m because $V_t(1, 0, p_t; \mathbf{d})$ is convex in d_m for all $d_m \geq 0$ (proved in Lemma 3). In fact, $\mathsf{E}_t\Delta_{t+1}$ decreases in d_m first and then increases above zero. Therefore, there exists a threshold debt level, denoted as D_t and also marked in Figure 4, such that the effect of bankruptcy cost dominates when $d_m \in (0, D_t)$, while the limited liability effect dominates when $d_m > D_t$. We formally state this result in the following proposition.

Proposition 5 *Suppose the debt obligation requires a single payment d_m at the end of period $m < T$, and $\mathsf{Pr}\{P_t < R_t^o\} > 0$ for all $t \leq m$.*

(i) In period $t \leq m$, there exists a unique threshold debt level $D_t > 0$ (which may depend on P_t), such that $\mathsf{E}_t\Delta_{t+1} > 0$ and strictly increases in d_m if $d_m > D_t$, and $\mathsf{E}_t\Delta_{t+1} < 0$ if $d_m \in (0, D_t)$.

(ii) If $d_m \geq D_t$, then $R_t^{(1)} = R_t^{(2)} \geq R_t^o$, and $R_t^{(1)}$ and $R_t^{(2)}$ increase in d_m. If $d_m \in (0, D_t)$, then $R_t^{(1)}$ increases in d_m and $R_t^{(2)} = R_t^o$ is invariant with respect to d_m. The optimal first-selling time τ (defined in Proposition 4) increases in d_m for $d_m > 0$.

(iii) If prices P_t, $t \in \mathcal{T}$, are independent (not necessarily i.i.d.), then $R_t^{(1)}$ and $R_t^{(2)}$ are deterministic, and the threshold debt level is a constant: $D_t \equiv R_m^o$ for $1 \leq t \leq m$, where R_m^o is the reservation price in period m for the no-debt case (defined in (4)).

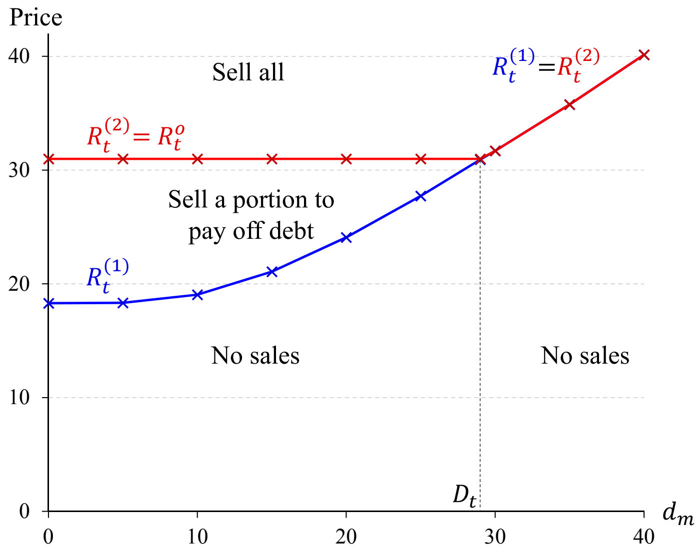
If $d_m \geq R_m^o$, then $R_t^{(1)} = R_t^{(2)} \geq R_t^o$, $1 \leq t \leq m$, and $\tau \geq \tau^o$, where equalities hold if $d_m = R_m^o$.

If $d_m < R_m^o$, then $R_t^{(1)} < R_t^{(2)} = R_t^o$, $1 \leq t \leq m$, and $\tau \leq \tau^o$.

Proposition 5 (i) and (ii) imply that, if the leverage is large ($d_m > D_t$), then the limited liability effect dominates and drives the seller's optimal policy—the seller should delay sales (i.e., the critical prices are higher than the no-debt case) and the seller will either sell the entire asset or sell nothing in each period; see Figure 5, where $R_t^{(1)} = R_t^{(2)}$ for $d_m \geq D_t$. The critical prices increase in the debt level, which means that the seller will delay selling the asset even longer.

Figure 5: Effect of debt level on critical prices: $T = 10$, $m = 7$

P_t's are i.i.d. lognormal and $\log(P_t) \sim \mathcal{N}(3, 0.5)$, $\rho = 0.98$, showing critical prices in period $t = 6$



If the leverage is small ($d_m < D_t$), the effect of bankruptcy cost is stronger than the limited liability effect. Bankruptcy cost incentivizes the seller to sell a portion of the asset to pay off the debt so that the firm becomes free of the constraint, as illustrated in Figure 5. As the debt level d_m reduces from D_t, $R_t^{(1)}$ decreases (while $R_t^{(2)}$ remains constant), hence the price region for a partial sale expands. This implies that the seller is more likely to sell a portion of the asset earlier for paying a smaller debt, contradicting the intuition that a larger debt will force the seller to sell prematurely. It is a smaller debt that incentivizes early sales.

The above results are in stark contrast with the upward (posted) price distortion found in Besbes et al. (2018). The effect of bankruptcy cost, when it dominates, reduces the reservation price below the revenue-maximizing reservation price. In a special situation when $d_m = D_t$, $R_t^{(1)} = R_t^{(2)} = R_t^o$, which means ithat the optimal policy under the debt coincides with the revenue-maximizing policy. This is because the incentive to delay sales due to limited liability balances with the incentive to expedite sales due to the bankruptcy cost. We note that the effect of bankruptcy cost can also exist in the posted-price setting: A model similar to the debt amortization model in Besbes et al. (2018) but including revenue loss after bankruptcy can also produce downward price distortion.

Note that D_t generally depends on P_t, while the debt level d_m is predetermined in period 0. Hence, it is possible that in some periods there are two distinct critical prices ($R_t^{(1)} < R_t^{(2)}$) while in some other periods there is only one critical price ($R_t^{(1)} = R_t^{(2)}$). Interestingly, when prices are independently (not necessarily identically) distributed, D_t becomes a constant and equal to R_m^o, as stated in Proposition 5(iii). Consequently, the optimal policy is characterized by either two distinct reservation prices (when $d_m < R_m^o$) or a single reservation price (when $d_m \geq R_m^o$).

3.4 General Debt Payment

We now generalize the analysis to settings where the debt financing agreement requires multiple payments. Specifically, the debt obligation requires k installments paid in periods $m_1, m_2, \ldots, m_k$ ($1 \leq m_1 < \cdots < m_k \leq T$), where $m_k \equiv m$ is the last payment period. That is, the debt payment schedule $\mathbf{d}$ has $d_t > 0$, for $t \in \{m_1, \ldots, m_k\}$, and $d_t = 0$ otherwise.

We show that, if the seller makes a sale in any period, the seller should sell either all of the remaining asset or an amount that exactly covers the debt payments required for the current and next several consecutive periods. We prove this optimal policy structure for the first sale in Proposition 6 and then prove the structural equivalence between the first sale and future sales in Proposition 7.

As before, let τ denote the time of the first sale, taking values in $\{1, \ldots, m_1, \infty\}$. Note that the seller must make the first sale on or before period m_1 to avoid bankruptcy.

Proposition 6 *Suppose the debt obligation requires k installments paid in periods $m_1, m_2, \ldots, m_k$ ($1 \leq m_1 < \cdots < m_k \leq T$).*
(i) There exists a series of critical prices $\{R_t^\dagger : 1 \leq t \leq m_1\}$, such that the first selling time $\tau = \inf\{t : P_t \geq R_t^\dagger, 1 \leq t \leq m_1\}$. If $P_t < R_t^\dagger$ for all $t \leq m_1$ (i.e., $\tau = \infty$), then the seller bankrupts at the end of period m_1.
(ii) If $\tau \leq m_1$, the optimal quantity to sell is $q_\tau^* \in \{1\} \cup \left\{ \sum_{i=1}^{j} \rho^{m_i - \tau} d_{m_i}/p_\tau : j = 1, \ldots, k \right\}$.

Proposition 6 prescribes that the first sale is triggered by a critical price similar to that for the case of a single debt payment. When the selling opportunity occurs, the seller should either sell the entire asset or sell an amount that generates a revenue of $\sum_{i=1}^{j} \rho^{m_i - \tau} d_{m_i}$ to secure exactly the first j installments.

Proposition 7 *Suppose the first sale is made in period $\tau \leq m_1$ to pay off exactly j installments. Then, the remaining asset, $h \equiv 1 - \sum_{i=1}^{j} \rho^{m_i - \tau} d_{m_i}/p_\tau$, should be sold as follows:*
(i) If $j = k$ (all debt is paid off), sell h using the debt-free policy in Proposition 1;
(ii) If $j < k$, sell h by solving an asset-selling problem with $T - \tau$ periods and $k - j$ remaining debt payments. The size of the remaining asset is scaled to one and the asset price is scaled to $\widetilde{P}_t = hP_t$; the initial wealth of this $(T - \tau)$-period problem is zero.

Proposition 7 effectively decomposes the original problem into a sequence of structurally identical asset-selling problems, each of which covers a certain number of installments.

Under a single debt payment, Proposition 4 shows that the optimal policy is characterized by a control band. With multiple debt payments, one may conjecture two opposite forces influencing the optimal policy as the price changes. As p_t increases, the revenue earned from selling a given quantity increases. As the price becomes more favorable to the seller, this force will induce selling more and earn greater revenue. Doing so will enable the seller to raise more working capital (for multiple debt payments) and provide more flexibility for future sales as the seller becomes less constrained by the payment schedule. On the other hand, Proposition 6(ii) implies that the optimal sales quantity is not a continuous function in price. In fact, the sales quantity is from a finite set of candidates, $q_t^* = \sum_{i=1}^{j} \rho^{m_i - t} d_{m_i}/p_t$, each representing the amount that is equal to a partial sum of current and future payments. As the price increases, it suffices to sell less to cover the same debt payments (i.e., the quantity is an inverse function of p_t) and keep more assets to wait for a higher price in the future. Consequently, depending on which of the two forces is stronger, the number of installments that the seller secures through a partial sale can non-monotonically change in price.

In general, characterizing the complete structure of optimal policy is difficult. This is, in part, because the value function is only convex in q within $\sum_{i=1}^{j} \rho^{m_i - t} d_{m_i}/p_t$ and $\sum_{i=1}^{j+1} \rho^{m_i - t} d_{m_i}/p_t$. As p_t increases, the amount of asset that needs to be sold decreases non-linearly in p_t. In order to evaluate the exact impact of a price change, we need to evaluate how multiple convex functions change in p_t at extreme points. This depends on payment structure, discount factor, working capital level, and, most importantly, price process. Numerically, we find that the impact of limited liability and bankruptcy cost on the optimal selling policy is consistent with the previous finding. We present the numerical results together with the capacitated selling case in §4.2.3.

4. Asset Selling with Capacity Constraint

In this section, we examine the asset-selling policies under capacity constraint, i.e., the maximum amount sold in one period is limited by $\ell < 1$. We show that when the debt requires a single payment at the end of the horizon, the asset-selling problem is equivalent to the problem of selling multiple non-divisible units. We characterize how the debt affects optimal policies and compare them with the policies in the previous section and in the literature.

4.1 Debt-Free Asset-Selling Policy

When there is no debt, the seller balances the revenue in the current period with the expected value of asset in the future when deciding the sales quantity. Proposition 8 shows the optimal policy.

Proposition 8 *Suppose the selling capacity $\ell < 1$ and there is no debt. Then, it is optimal to divide* *the asset into* $n \stackrel{\text{def}}{=} \lceil \frac{1}{\ell} \rceil$ *pieces,*[2] *with* $n-1$ *pieces of size* ℓ *and a remainder of size* $r \equiv 1-(n-1)\ell$. *The optimal sequential selling policy is characterized by critical prices* $\{R^o_{t,i} : t \in \mathcal{T},\ i = 1, \ldots, n\}$ *with* $R^o_{t,i}$ *increasing in* i *and representing the critical price for the* i*-th sale.*

(i) For given $i \in \{1, \ldots, n\}$*, suppose that the seller has sold* $i-1$ *times before period* t*. In period* t*,*

(a) If the remainder r *has been previously sold, it is optimal to sell* ℓ *if* $P_t \geq R^o_{t,i}$ *and sell nothing otherwise;*

(b) If the remainder r *has not been previously sold, it is optimal to sell* ℓ *if* $P_t \geq R^o_{t,i+1}$*, sell* r *if* $R^o_{t,i} \leq P_t < R^o_{t,i+1}$*, and sell nothing otherwise.*

(ii) For $i = 1, \ldots n-1$*, the critical prices satisfy* $R^o_{T,i} = 0$ *and* $R^o_{t,i} = \rho \mathsf{E}_t \text{median}\{P_{t+1}, R^o_{t+1,i}, R^o_{t+1,i+1}\}$ *for* $t = 1, \ldots, T-1$*. Furthermore,* $R^o_{t,n} = R^o_t$ *for all* t*, i.e., the critical price for the last sale is the same as the critical price for the case without capacity constraint.*

Proposition 8(i) reveals that the asset-selling problem with capacity constraint is equivalent to selling n non-divisible assets, where n is the least number of sales the seller must make to sell the entire asset. The optimal policy features a sequence of critical prices that characterize when and how much of the asset is sold for each of the n sales. In addition, when the seller has not sold the remainder r, a control band with two critical prices are in play: If the price is moderately favorable, then sell the remainder; if the price is more favorable, sell the maximum quantity ℓ.

Proposition 8(ii) details the properties of the critical prices. Note that because $R^o_{t,n} = R^o_t$, it satisfies the relation proved in Proposition 1: $R^o_{t,n} = \rho \mathsf{E}_t \max\{P_{t+1}, R^o_{t+1,n}\}$, which means that $R^o_{t,n}$ is the expected best selling price after period t. The median relation in part (ii) can be written as $R^o_{t,i} = \rho \mathsf{E}_t \min\big\{\max\{P_{t+1}, R^o_{t+1,i}\}, R^o_{t+1,i+1}\big\}$, which means that $R^o_{t,i}$ is the expected $(n+1-i)$-th best selling price after t. Indeed, the first sale is triggered by $P_t \geq R^o_{t,1}$, i.e., the current price exceeds the expected n-th best selling price in the future.

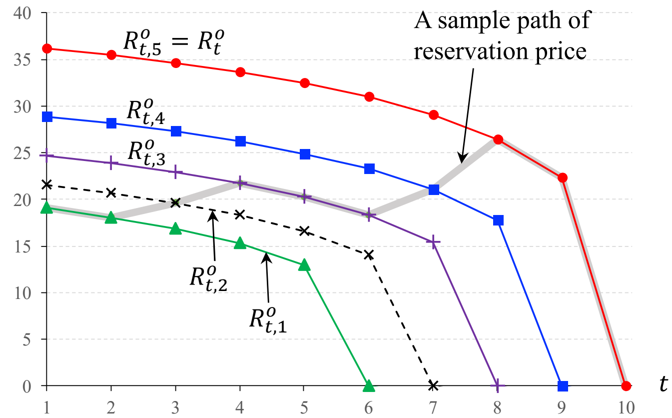
Note that, if prices are i.i.d., the above problem of selling n portions of the asset is essentially a stochastic assignment problem first studied by Derman et al. (1975). We generalize their results to Markov price process and further derive the relations between critical prices in the simplest form. This generation is possible because the value function $\rho \mathsf{E}_t U_{t+1}(x_{t+1}, P_{t+1})$ is concave and piecewise linear in x_{t+1}, and the slopes for the n segments $(0, \ell], (\ell, 2\ell], \ldots, ((n-2)\ell, (n-1)\ell]$, and $((n-1)\ell, 1]$ are $R^o_{t,n} \geq R^o_{t,n-1} \geq \cdots \geq R^o_{t,1}$, respectively (see the proof of Proposition 8).

When prices are i.i.d., $R_{t,i}$'s are deterministic reservation prices, illustrated in Figure 6. If a sale does (does not) occur in a period, the reservation price increases (decreases) in the next period; see the sample path of reservation price in Figure 6. This resembles the posted-price changes in Gallego

[2] $\lceil x \rceil$ is the ceiling function that gives the smallest integer greater than or equal to x.

Figure 6: Critical prices under selling capacity $\ell = 0.2$, without debt

P_t's are i.i.d. and $\log(P_t) \sim \mathcal{N}(3, 0.5)$, $\rho = 0.98$, $R_{t,i}$ is the reservation price for the i-th sale



and van Ryzin (1994). The key difference is that posted prices are no lower than the one-period revenue-maximizing price and there may be unsold items at the end of the horizon, whereas the reservation price drops to zero whenever the number of remaining periods is equal to the number of unsold units.

4.2 Asset Selling Under Debt and Capacity Constraint

As we did in §3, we examine the case with a single payment at the end of planning horizon ($m = T$), and extend our study to a single payment at $m < T$ as well as general cases. Because capacity constraint introduces additional analytical difficulty, the case with general debt schedule is not tractable. However, our numerical analysis will demonstrate that even under capacity constraints, limited liability and bankruptcy cost remain to drive the optimal asset-selling policy.

4.2.1 Single Debt Payment at $m = T$

Parallel to §3.2, we analyze the selling policy when the debt requires a single payment at $m = T$. Lemma 5 generalizes Lemma 2, and Proposition 9 characterizes the optimal policy.

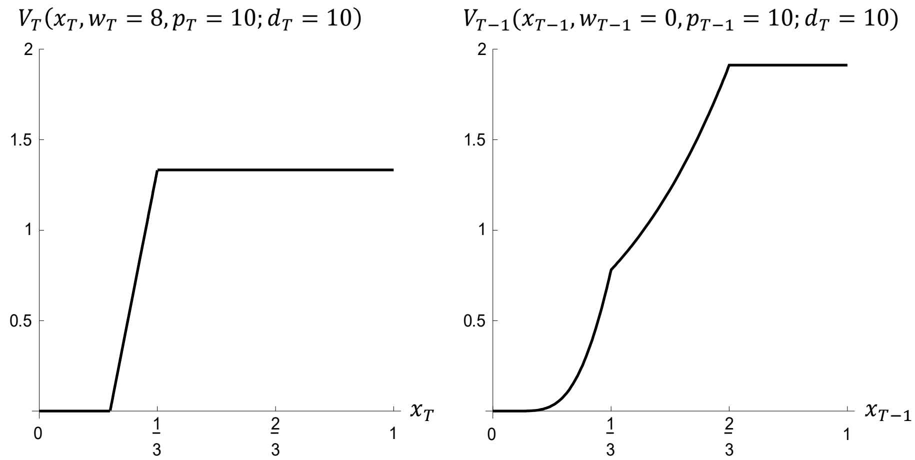
Lemma 5 *Suppose $d_t = 0$ for $t = 1, \ldots, T-1$ and $d_T > 0$. Then, for $t \in \mathcal{T}$, $V_t(x_t, w_t, p_t; \mathbf{d})$ is convex in (x_t, w_t) in each of the n regions: $x_t \in \big[(j-1)\ell, \min\{j\ell, 1\}\big]$, $j = 1, 2, \ldots, n = \left\lceil \frac{1}{\ell} \right\rceil$.*

Figure 7 shows the equity value as a function of inventory in the last two periods under selling capacity $\ell = 1/3$. The piecewise convexity is evident. In addition, notice that $V_{T-1}(\frac{1}{3}) - V_{T-1}(0) \leq V_{T-1}(\frac{2}{3}) - V_{T-1}(\frac{1}{3})$ and $V_{T-1}(\frac{2}{3}) - V_{T-1}(\frac{1}{3}) \geq V_{T-1}(1) - V_{T-1}(\frac{2}{3})$, which means that inventories in discrete units of size $\frac{1}{3}$ may exhibit increasing marginal values (due to limited liability) or decreasing marginal values (due to limited capacity).

Proposition 9 *Suppose $d_t = 0$ for $t = 1, \ldots, T-1$ and $d_T > 0$. Then, it is optimal to divide the*

Figure 7: Piecewise convexity of the value function

$\ell = 1/3$, P_t's are i.i.d. and $\log(P_t) \sim \mathcal{N}(3, 0.5)$, $\rho = 1$



asset into $n = \lceil \frac{1}{\ell} \rceil$ pieces, with $n-1$ pieces of size ℓ and a remainder of size $r \equiv 1-(n-1)\ell$. There exists a series of critical prices $\{R_t^c : t \in \mathcal{T}\}$ such that

(i) It is optimal to make the first sale in period $\tau^c \overset{\text{def}}{=} \inf\{t : P_t \geq R_t^c, t \in \mathcal{T}\}$. If $P_t < R_t^c$ for all $t \in \mathcal{T}$ (i.e., $\tau^c = \infty$), then the seller bankrupts at the end of period T.

(ii) If $\tau^c < T$, then in period $\tau^c + 1$, the seller faces a problem that is structurally identical to the original problem, with $T - \tau^c$ periods and a reduced debt $d'_T = \left(d_T - q^*_{\tau^c} p_{\tau^c} / \rho^{T-\tau^c}\right)^+$, where $q^*_{\tau^c} \in \{\ell, r\}$ is the optimal first sales quantity.

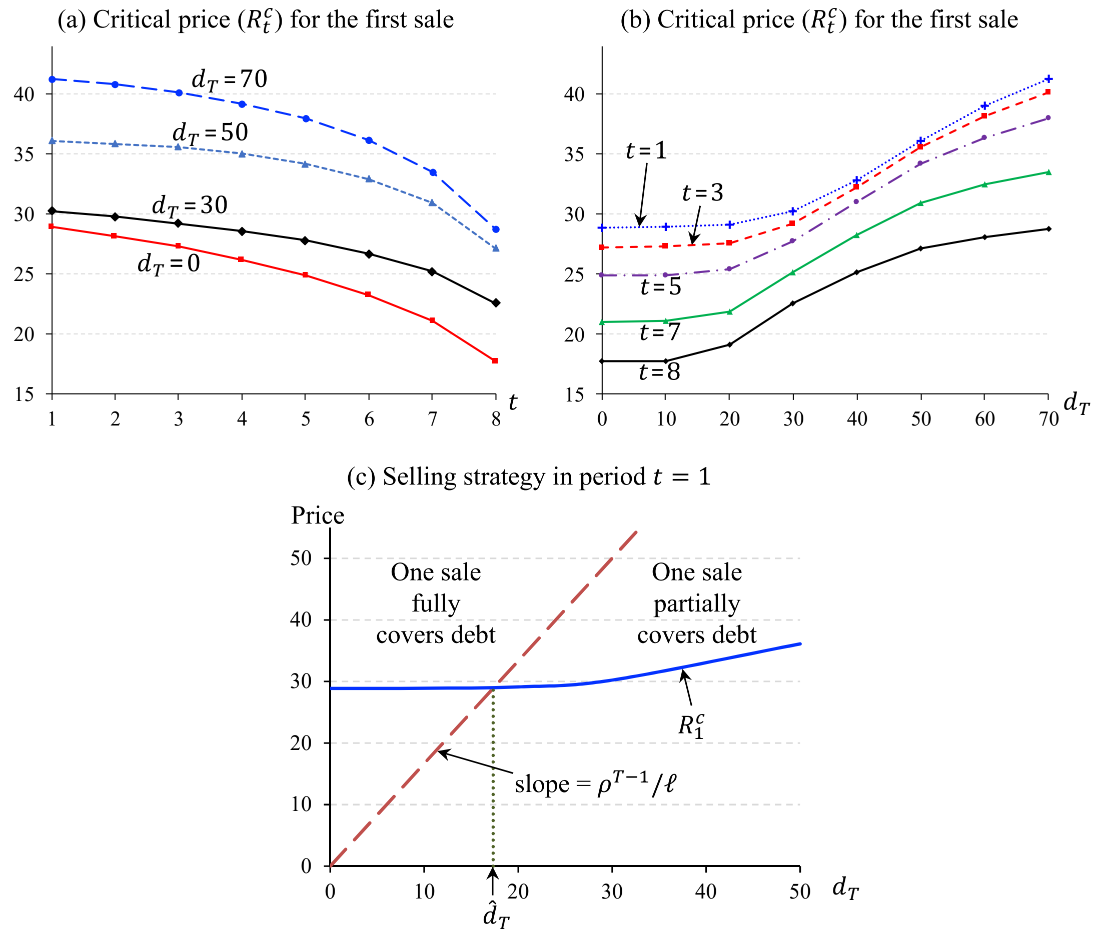
Proposition 9 shows that the seller divides the asset into multiples of ℓ (plus a remainder) for sale; this division is exactly the same as in the no-debt case. The first sale occurs when the price exceeds a critical price R_t^c. We next examine how the debt level affects this critical price.

Besbes et al. (2018) find that the upward (posted) price distortion increases over time. In the absence of selling capacity, we also find that (critical) price distortion increases over time if $m = T$ (see Figure 1(a)). However, in the presence of selling capacity, the distortions exhibit intricate patterns. Consider a case with capacity $\ell = 0.5$, under which the asset is divided into two halves for sale. Figure 8(a) shows that, only at low debt levels, the distortion increases over time (compare the critical prices for $d_T = 0$ and $d_T = 30$). At high debt levels, the (marginal) distortion is decreasing over time (compare the critical prices for $d_T = 50$ and $d_T = 70$). Panel (b) illustrates this pattern using more refined debt levels and confirms that as d_T increases, the time period with the largest marginal distortion on critical price shifts from late in the horizon to early in the horizon.

The above pattern can be explained by understanding two distinct forces on price distortion. First, as debt maturity nears, the limited liability has a stronger upward effect on the critical price. This force leads to increasing distortion over time. Second, because the selling capacity prevents

Figure 8: Optimal policy for the first sale with capacity $\ell = 0.5$

P_t's are i.i.d. lognormal and $\log(P_t) \sim \mathcal{N}(3, 0.5)$, $\rho = 0.98$, $m = T = 10$

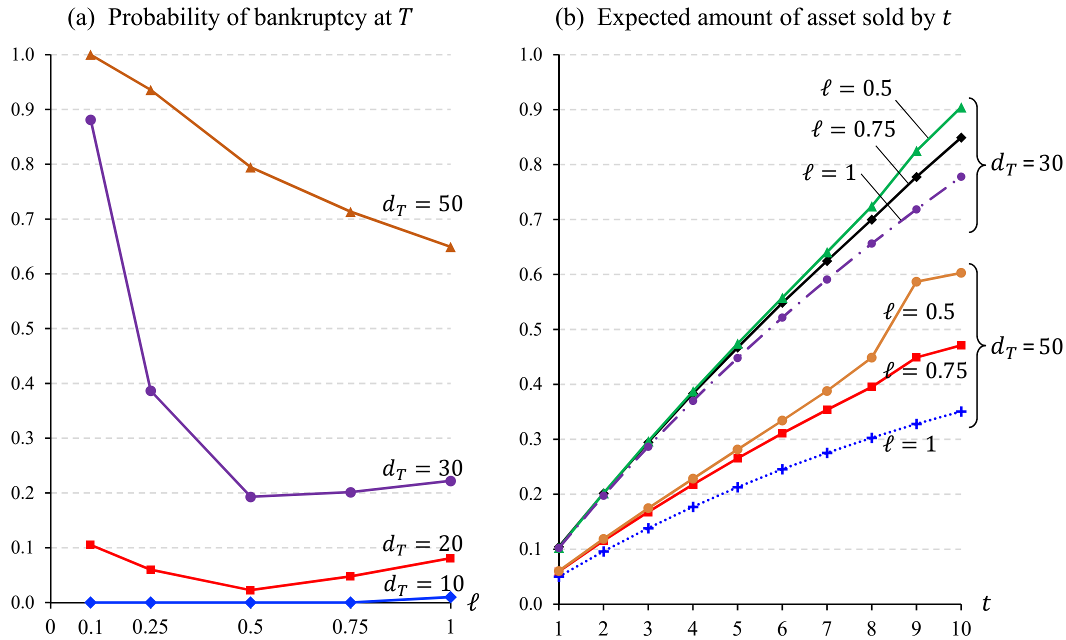


the asset from being sold all in one period, when the selling time is running out and no sale has yet occurred, there is a downward pressure on the critical price. (Indeed, the critical price drops to zero in period 9 to ensure the first sale occurs.) Under a high debt level, the second force becomes particularly strong when the first force tries to raise the critical price. The net effect is reduced critical price distortion over time.

We now examine how the debt is covered for the case of $\ell = 0.5$. If the revenue from the first sale covers the debt, the second sale will follow the debt-free selling policy in Proposition 1, otherwise the second sale will follow Proposition 3. Figure 8(c) shows that under a relatively small debt ($d_T < \hat{d}_T$ in the figure), the seller may sell nothing even if selling some of the asset can cover the debt. Under a relatively large debt ($d_T > \hat{d}_T$), however, the firm may sell part of the asset even if it is insufficient to cover the debt. Besbes et al. (2018) find that the seller uses a single sale to cover a small debt and uses multiple sales to cover a large debt. Our result differs in that for any given debt, it is always possible to cover the debt with one sale as long as the realized price is high enough.

Figure 9: Impact of selling capacity on bankruptcy probability and expected sales

P_t's are i.i.d. lognormal and $\log(P_t) \sim \mathcal{N}(3, 0.5)$, $\rho = 0.98$, $m = T = 10$



Finally, we examine the impact of capacity constraint on bankruptcy risk. The conventional wisdom is that an operationally constrained firm has a higher default risk. Figure 9(a) shows that, at a high debt level (e.g., $d_T = 50$), a tighter capacity constraint (lower ℓ) indeed increases the probability of bankruptcy. Interestingly, for medium debt levels ($d_T = 20$ and 30), when capacity tightens (ℓ decreases), the bankruptcy probability first decreases and then increases. This is because when a capacity constraint is present but not too tight, the optimal policy would encourage selling the asset earlier while still capturing high prices, leading to a lower default risk. Figure 9(b) confirms that the asset is indeed sold earlier when capacity constraint tightens. When the capacity constraint becomes very tight, however, the seller may have to sell the asset at adverse prices, resulting in a higher probability of falling short of covering the debt. Lastly, for very low debt levels ($d_T = 10$), without capacity constraint, the bankruptcy probability is 0.01, whereas a capacity constraint (regardless how tight) effectively removes the bankruptcy risk. This is because the capacity constraint induces early sales, which can easily cover a small debt. The above finding suggests that when neither the debt obligation nor selling capacity is stringent, additional selling capacity only encourages the firm to engage in riskier strategies. Iancu et al. (2017) also find a negative effect of operating flexibility under debt.

4.2.2 Single Debt Payment at $m < T$

With a capacity constraint and a debt maturing before the end of the horizon, the value function $V_t(x_t, w_t, p_t; \mathbf{d})$ may not be continuous in w_t and p_t. To see the discontinuity, suppose in period $t < T$, the seller has inventory $x_t \in (\ell, 1]$ and consider the cases in (3): if $w_t + p_t\ell < d_t$, bankruptcy occurs and $V_t(x_t, w_t, p_t; \mathbf{d}) = 0$, whereas if $w_t + p_t\ell = d_t$, the seller can survive by selling at capacity ℓ, leading to $V_t(x_t, w_t, p_t; \mathbf{d}) = \rho \mathsf{E}_t V_{t+1}(x_t - \ell, 0, P_{t+1}; \mathbf{d}) > 0$, where $x_t - \ell > 0$ and $w_{t+1} = 0$. Hence, the value function is discontinuous at $w_t + p_t\ell = d_t$ when $x_t \in (\ell, 1]$.

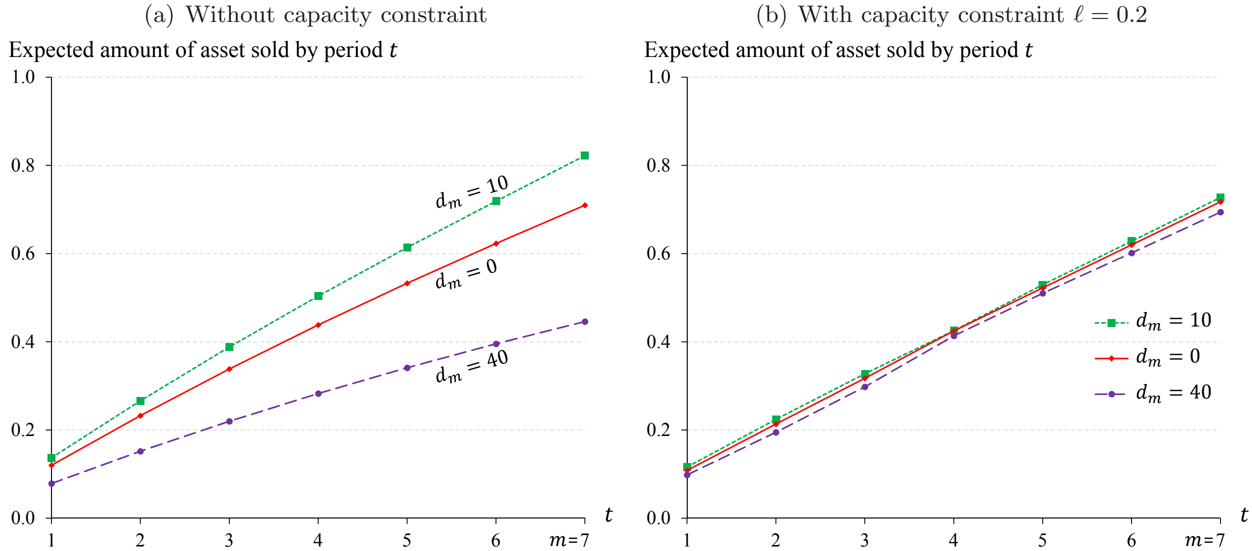
In addition, recall that, when there is no capacity constraint, the value function is convex or piecewise convex (Lemmas 2 and 3). However, §4.1 shows that the capacity constraint alone leads to a *concave* value function. With both debt and capacity constraints in period t, the value function is neither concave or convex. Hence, debt and capacity constraints render the problem analytically intractable in general. Therefore, we resort to numerical analysis to analyze the selling policy, focusing on the essential tradeoff between the benefit of limited liability and the value loss due to bankruptcy cost.

We will show that the two effects resulting from limited liability and bankruptcy cost, respectively, still exist when capacity constraint is present. Figure 10 compares the expected cumulative sales with and without the capacity constraint. Panel (a) considers the same setting as in Figure 5 without the capacity constraint. In the small debt case ($d_m = 10$), the expected cumulative sales in any $t \leq m$ exceeds the sales amount in the debt-free case, which means sales are expedited, whereas in the large debt case ($d_m = 40$), the sales are delayed.

Figure 10: Expected amount of asset sold by period t: single debt payment at $m = 7$

P_t's are i.i.d. lognormal and $\log(P_t) \sim \mathcal{N}(3, 0.5)$, $\rho = 0.98$, $T = 10$

(a) Without capacity constraint (b) With capacity constraint $\ell = 0.2$



With the capacity constraint of $\ell = 0.2$, panel (b) reveals that the presence of bankruptcy cost expedites selling under a small debt. On the other hand, under a large debt, the sale is delayed due to the limited liability effect. However, the magnitude of expedition or delay is much smaller compared to that in panel (a). Intuitively, the capacity limit $\ell = 0.2$ means that the asset needs to be sold over at least five periods within a 10-period planning horizon. This constraint significantly reduces the flexibility in expediting or delaying sales.

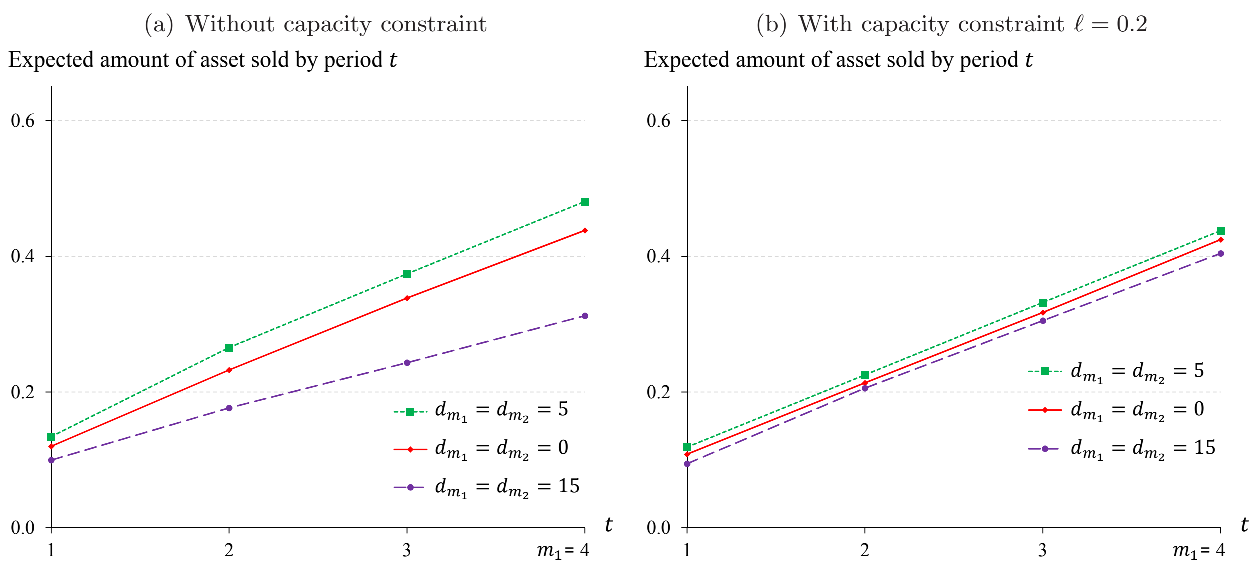
4.2.3 General Debt Payment

We further study more general debt payment schemes. Recall from §3.4 that the general debt payment cases are analytically difficult to solve; moreover, as discussed in §4.2.2, the capacity constraint renders the value function discontinuous. Despite these complications, we can numerically demonstrate that the key insights still hold for the general case.

Specifically, we consider the case with two equal debt payments at $m_1 = 4$ and $m_2 = 7$. Figure 11 presents the expected cumulative sales before the first installment is due. Consistent with all the previous results, asset selling is expedited under small debts and is delayed under large debts. However, as the capacity constraint tightens, these effects are subdued.

Figure 11: Expected amount of asset sold by period t: two debt payments at $m_1 = 4$ and $m_2 = 7$

P_t's are i.i.d. lognormal and $\log(P_t) \sim \mathcal{N}(3, 0.5)$, $\rho = 0.98$, $T = 10$



4.2.4 Asset-Selling Example under General Price Process: Selling Natural Gas

In this section, we apply our model to the industrial setting of selling natural gas. We employ a commodity price model with parameters estimated from the natural gas price data.

Suppose the asset is 10^6 MBtu (million British thermal units) of natural gas. Following the literature on commodity asset pricing (Schwartz 1997, Jaillet et al. 2004), we model the logarithmic price $\log P_t$ as an Ornstein-Uhlenbeck process in continuous time:

$$d \log P_t = \kappa(\mu - \log P_t)dt + \overline{\sigma} dZ_t. \tag{6}$$

where μ is long-term average level, κ is the mean-reversion rate, $\overline{\sigma}$ is volatility and Z is a standard Brownian motion. We calibrate model (6) using weekly Henry Hub natural gas spot price data, which is obtained from the Energy Information Administration and covers 157 observations from May 1, 2015 to April 30, 2018 (www.eia.gov/dnav/ng/hist/rngwhhdW.htm). We use the maximum likelihood estimation (see Tang and Chen 2009 and Sørensen 2004) to estimate the parameters and obtain $\kappa = 0.129$, $\mu = 0.994$, and $\overline{\sigma} = 0.097$.

A discrete-time sample of the price in (6) is an AR(1) process:

$$\log(P_{t+1}) = \eta + \beta \log(P_t) + \epsilon_t, \tag{7}$$

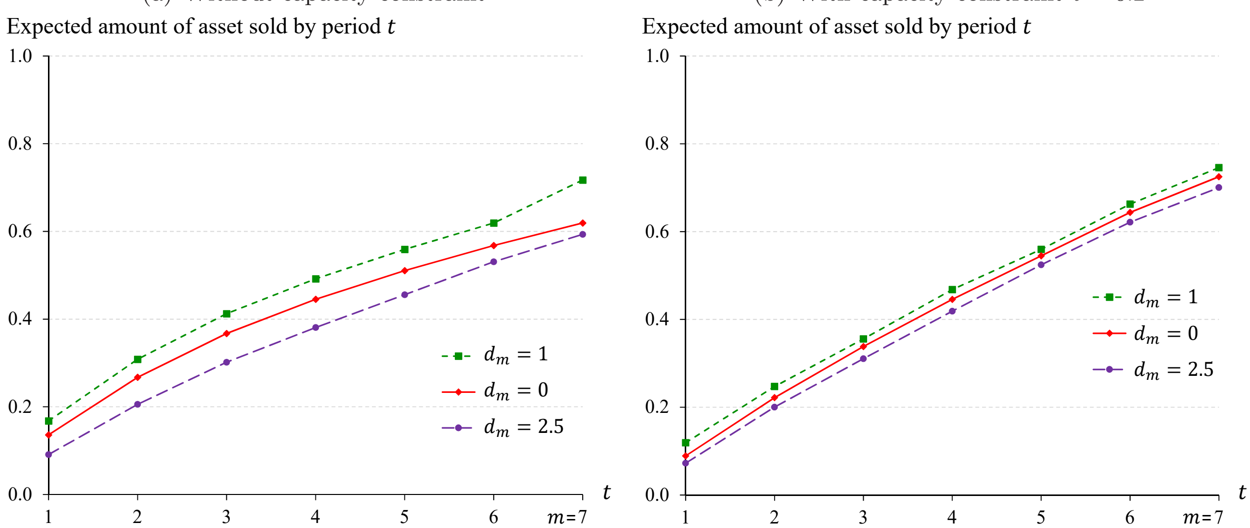
where each period t represents a $\Delta t = 4$ weeks, $\beta = e^{-\kappa\Delta t} = 0.879$, $\eta = (1-\beta)\mu = 0.120$, and $\epsilon_t \sim N(0, \sigma^2)$ is the i.i.d. random shock with $\sigma = \overline{\sigma}\sqrt{\frac{1-e^{-2\kappa\Delta t}}{2\kappa}} = 0.153$.

The expected sales over time are illustrated in Figure 12 under three debt levels: \$0, \$1 million, and \$2.5 million, with and without selling capacity. The results are reassuring. We see that the natural gas sales can be delayed (or advanced) if limited liability (or bankruptcy cost) effect is dominant. Observe that the attenuation effect of capacity constraint is smaller in Figure 12 compared to Figure 10. This is because the benefit of waiting for a more favorable selling price is lower under

Figure 12: Example of selling natural gas: single debt payment at $m = 7$

$\log(P_t)$ follows the process in (7) with P_1 having the stationary distribution, $\rho = 0.99$, $T = 10$

(a) Without capacity constraint (b) With capacity constraint $\ell = 0.2$



sticky prices (e.g., Markov price process in (7)) than under independent prices. Similarly, the effect of bankruptcy cost is weaker when the prices are sticky. Thus, the attenuation effect of capacity constraint is less pronounced.

5. Extension and Conclusions

5.1 Asset Liquidation

The model in (2)-(3) assumes that when the firm is unable to make a scheduled debt repayment, it goes into bankruptcy and the payoff is zero. Although zero payoff upon bankruptcy is a common assumption in the literature, it is an extreme case. In practice, if the firm files for bankruptcy under Chapter 7, the remaining asset is liquidated and the revenue from liquidation sales will be distributed to the creditors up to the total unpaid debt. The residual revenue will be returned to the firm's shareholders. In this section, we extend our basic model to include the liquidation process and compare it with the results in the previous sections.

Formally, suppose the firm goes bankrupt in period t with unpaid debt $(d_t^r, d_{t+1}, \ldots, d_m)$ and unsold asset x_{t+1}. The remaining asset x_{t+1} will be sold to maximize the expected revenue over a liquidation period from $t+1$ up to $t+L < T$, where L is the length of the liquidation period. We assume that the remaining asset x_{t+1} can still be sold at the market price, but the liquidation period is typically short and, therefore, the expected revenue from liquidation sales is lower than if the remaining asset were sold over the remaining planning horizon. This reduction in asset value is a form of the indirect bankruptcy cost.

Liquidating x_{t+1} follows a revenue-maximization policy that is structurally the same as the policy for (1) but with a different discount rate. For simplicity, we assume no discounting for the revenues from liquidation sales. Let $\{q_s^B : s = t+1, \ldots, t+L\}$ be a sequence of random variables denoting the revenue-maximization selling quantities over L periods of liquidation sales.

The total liquidation revenue $\sum_{s=t+1}^{t+L} P_s q_s^B$ will be used first to pay the remaining debt $d_t^r + \sum_{s=t+1}^{m} d_s$. The firm is expected to obtain the following residual payoff at the end of period $t+L$:

$$\Omega_{t+1}(x_{t+1}, d_t^r; \mathbf{d}) = \mathsf{E}_t \left[\sum_{s=t+1}^{t+L} P_s q_s^B - d_t^r - \sum_{s=t+1}^{m} d_s \right]^+. \tag{8}$$

We can rewrite the firm's optimization problem in (2)-(3) as follows:

$$\begin{aligned} V_m(x_m, w_m, p_m; \mathbf{d}) = \max \Big\{ & \max_{q_m \in [0, \min(x_m, \ell)] \cap [\underline{q}_m, \infty)} p_m q_m + w_m - d_m + \rho \mathsf{E}_m U_{m+1}(x_m - q_m, P_{m+1}), \\ & \max_{q_m \in [0, \min(x_m, \ell, \underline{q}_m)]} \rho^L \Omega_{m+1}(x_m - q_m, d_m - p_m q_m - w_m; \mathbf{d}) \Big\}, \end{aligned} \tag{9}$$

$$V_t(x_t, w_t, p_t; \mathbf{d}) = \max\Big\{ \max_{q_t \in [0, \min(x_t, \ell)] \cap [\underline{q_t}, \infty)} \rho \mathsf{E}_t V_{t+1}\big(x_t - q_t,\ \rho^{-1}(w_t + p_t q_t - d_t),\ P_{t+1}; \mathbf{d}\big),$$
$$\max_{q_t \in [0, \min(x_t, \ell, \underline{q_t})]} \rho^L \Omega_{t+1}(x_t - q_t, d_t - p_t q_t - w_t; \mathbf{d})\Big\}, \quad \text{for } t < m. \qquad (10)$$

Note that if additional direct and indirect bankruptcy costs (such as administrative cost, legal fees, and reputation loss) are included in (8) so that the residual payoff to the firm is zero: $\Omega_{t+1} = 0$, then (9)-(10) are equivalent to (2)-(3).

In (9)-(10), the inner problem of choosing $q_t \in [0, \min(x_t, \ell)] \cap [\underline{q_t}, \infty)$ finds the value of the firm if surviving,[3] while the inner problem of choosing $q_t \in [0, \min(x_t, \ell, \underline{q_t})]$ finds the value of the firm if bankrupt. The firm chooses the greater of the two maximum values. Therefore, the optimal selling strategies for (9)-(10) involves a new feature: the firm may choose to go bankrupt even if it can survive. This is because in some situations, survival brings very low payoff, e.g., when $\underline{q_t} = x_t \leq \ell$, the firm has to sell all the remaining asset in order to survive, yielding zero payoff, whereas going bankrupt may yield a positive expected payoff to the firm.

Figure 13: Optimal asset-selling policy under a single debt payment

Same parameters as in Figure 3, liquidation period $L = 1$, debt $d_m = 10$

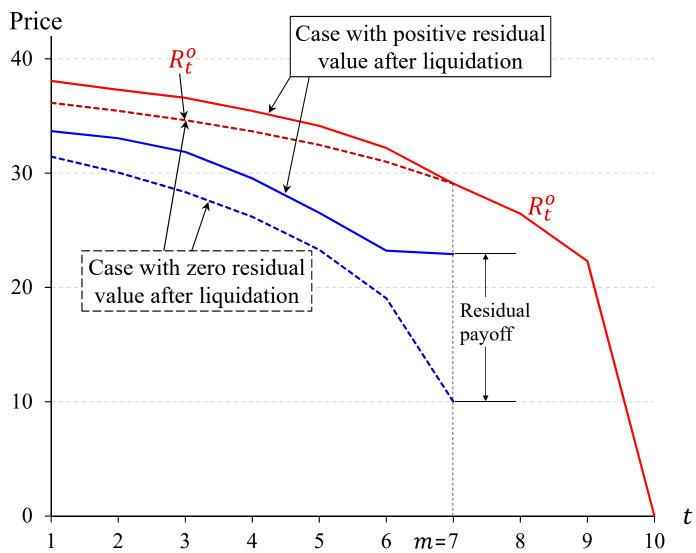


Figure 13 compares the optimal policy under zero residual payoff with the optimal policy under a positive residual payoff. Specifically, if the firm goes bankrupt in period $m = 7$, the asset is liquidated in period 8. The dashed curves in Figure 13 correspond to the critical prices under zero residual payoff, also shown in Figure 3. With a positive residual payoff, both critical prices are higher. This is because when the effect of bankruptcy cost is weakened by the presence of residual payoff,

[3] If $\underline{q_t} > \min(x_t, \ell)$, then the feasible set $[0, \min(x_t, \ell)] \cap [\underline{q_t}, \infty) = \varnothing$, implying that the firm cannot survive and the maximum value is $-\infty$ by convention.

the limited liability effect becomes more prominent. Importantly, the policy structure identified in Proposition 4 is robust: when the price falls between the two critical prices (solid curves), the firm should sell a portion of the asset to cover its debt exactly. Furthermore, in this example the lower critical prices are consistently below R_t^o, the critical price without debt, indicating that part of the asset is sold earlier than the debt-free case.

At maturity, with no residual payoff, the lower critical price is $R_m^{(1)} = d_m$. With a residual payoff, however, this critical price is lifted by an amount that is exactly the residual payoff. This is because in period m, the firm will choose to go bankrupt if the expected liquidation value of the asset is higher than the ongoing selling price p_m.

5.2 Concluding Remarks

The classical asset-selling problem has seen applications in practice where one or several non-divisible assets are to be sold. This paper extends the classical asset-selling problem to incorporate several realistic features, including debt obligations, selling capacity constraints, and Markov price evolution. With these features, our study has broadened the scope of the classic asset-selling problem and laid the foundation for its application to more industrial contexts.

We show that the Markov nature of the price process does not qualitatively change the structure of the optimal asset-selling policies. The debt obligation, however, can structurally alter the optimal policy. The debt obligation introduces the new tradeoff between the benefit of limited liability and the value loss due to bankruptcy cost. This tradeoff results in a new selling strategy, either to secure the debt payment early to avoid costly bankruptcy or to delay the sale as the bankruptcy curbs the downside risk. When both capacity constraint and debt obligation are present, we show that the key effects derived from the uncapacitated case—small debts expedite sales while large debts delay sales—continue to hold, although these effects are attenuated by the capacity constraint.

References

Albright, S. C. 1974. Optimal sequential assignments with random arrival times. *Management Science* **21**(1) 60–67.

Albright, S. C. 1977. A bayesian approach to a generalized house selling problem. *Management Science* **24**(4) 432–440.

Ang, J. S., J. H. Chua, J. J. McConnell 1982. The administrative costs of corporate bankruptcy: A note. *The Journal of Finance* **37**(1) 219–226.

Arnold, M. A., S. A. Lippman 2001. The analytics of search with posted prices. *Economic Theory* **17**(2) 447–466.

Batabyal, A. A., S. J. Yoo 2005. Indivisibility and divisibility in land development over time and under uncertainty. *Journal of environmental management* **76**(2) 185–190.

Besbes, O., C. Maglaras 2012. Dynamic pricing with financial milestones: feedback-form policies. *Management Science* **58**(9) 1715–1731.

Besbes, O., D. A. Iancu, N. Trichakis 2018. Dynamic pricing under debt: Spiraling distortions and efficiency losses. *Management Science* **64**(10) 4572–4589.

Boyabatli, O., L. B. Toktay 2011. Stochastic capacity investment and flexible vs. dedicated technology choice in imperfect capital markets. *Management Science* **57**(12) 2163–2179.

Bris, A., I. Welch, N. Zhu 2006. The costs of bankruptcy: Chapter 7 liquidation versus Chapter 11 reorganization. *The journal of finance* **61**(3) 1253–1303.

Buzacott, J. A., R. Q. Zhang 2004. Inventory management with asset-based financing. *Management Science* **50**(9) 1274–1292.

Chod, J. 2017. Inventory, risk shifting, and trade credit. *Management Science* **63**(10) 3207–3225.

Chod, J., J. Zhou 2013. Resource flexibility and capital structure. *Management Science* **60**(3) 708–729.

Chun, Y. H., R. D. Plante, H. Schneider 2002. Buying and selling an asset over the finite time horizon: A non-parametric approach. *European Journal of Operational Research* **136**(1) 106–120.

Derman, C., G. J. Lieberman, S. M. Ross 1972. A sequential stochastic assignment problem. *Management Science* **18**(7) 349–355.

Derman, C., G. J. Lieberman, S. M. Ross 1975. A stochastic sequential allocation model. *Operations Research* **23**(6) 1120–1130.

Ee, M.-S. 2009. Asset-selling problem with an uncertain deadline, quitting offer, and search skipping option. *European Journal of Operational Research* **198**(1) 215–222.

Freeman, P. R. 1983. The secretary problem and its extensions: A review. *International Statistical Review* **51**(2) 189–206.

Gallego, G., G. van Ryzin 1994. Optimal dynamic pricing of inventories with stochastic demand over finite horizons. *Management science* **40**(8) 999–1020.

Gallien, J. 2006. Dynamic mechanism design for online commerce. *Operations Research* **54**(2) 291–310.

Gilbert, J. P., F. Mosteller 1966. Recognizing the maximum of a sequence. *Journal of the American Statistical Association* **61**(313) 35–73.

Iancu, D. A., N. Trichakis, G. Tsoukalas 2017. Is operating flexibility harmful under debt?. *Management Science* **63**(6) 1730–1761.

Jaillet, P., E. I. Ronn, S. Tompaidis 2004. Valuation of commodity-based swing options. *Management science* **50**(7) 909–921.

Karlin, S. 1962. Stochastic models and optimal policy for selling an asset. *In* K. J. Arrow, S. Karlin and H. Scarf eds. *Studies in Applied Probability and Management Science*. Stanford University Press. pp. 148–158.

Kouvelis, P., W. Zhao 2011. The newsvendor problem and price-only contract when bankruptcy costs exist. *Production and Operations Management* **20**(6) 921–936.

Kouvelis, P., W. Zhao 2012. Financing the newsvendor: Supplier vs. bank, and the structure of optimal trade credit contracts. *Operations Research* **60**(3) 566–580.

Levin, Y., J. McGill, M. Nediak 2008. Risk in revenue management and dynamic pricing. *Operations Research* **56**(2) 326–343.

Lippman, S. A., J. McCall 1976. The economics of job search: A survey. *Economic Inquiry* **14**(2) 155–189.

Myers, S. C. 1977. Determinants of corporate borrowing. *Journal of financial economics* **5**(2) 147–175.

Palley, A. B., M. Kremer 2014. Sequential search and learning from rank feedback: Theory and experimental evidence. *Management Science* **60**(10) 2525–2542.

Phillips, R. L. 2005. *Pricing and revenue optimization*. Stanford University Press.

Prastacos, G. P. 1983. Optimal sequential investment decisions under conditions of uncertainty. *Management Science* **29**(1) 118–134.

Pye, G. 1971. Minimax policies for selling an asset and dollar averaging. *Management Science* **17**(7) 379–393.

Rogerson, R., R. Shimer, R. Wright 2005. Search-theoretic models of the labor market: A survey. *Journal of economic literature* **43**(4) 959–988.

Rosenfield, D. B., R. D. Shapiro, D. A. Butler 1983. Optimal strategies for selling an asset. *Management Science* **29**(9) 1051–1061.

Rothschild, M. 1974. Searching for the lowest price when the distribution of prices is unknown. *Journal of Political Economy* **82**(4) 689–711.

Schwartz, E. S. 1997. The stochastic behavior of commodity prices: Implications for valuation and hedging. *The journal of finance* **52**(3) 923–973.

Sørensen, H. 2004. Parametric inference for diffusion processes observed at discrete points in time: a survey. *International Statistical Review* **72**(3) 337–354.

Stam, J. M., B. L. Dixon 2004. *Farmer bankruptcies and farm exits in the United States, 1899-2002*. Economic Research Service, U.S. Department of Agriculture. Agriculture Information Bulletin No. 788.

Stigler, G. J. 1961. The economics of information. *Journal of Political Economy* **69**(3) 213–225.

Su, X., S. A. Zenios 2005. Patient choice in kidney allocation: A sequential stochastic assignment model. *Operations Research* **53**(3) 443–455.

Tang, C. Y., S. X. Chen 2009. Parameter estimation and bias correction for diffusion processes. *Journal of Econometrics* **149**(1) 65–81.

Telser, L. G. 1973. Searching for the lowest price. *American Economic Review* **63**(2) 40–49.

Xu, X., J. R. Birge 2004. Joint production and financing decisions: Modeling and analysis. *Available at SSRN 652562*.

Yang, S. A., J. R. Birge, R. P. Parker 2015. The supply chain effects of bankruptcy. *Management Science* **61**(10) 2320–2338.

Online Appendix

Proof of Lemma 1: (i) We prove by induction. At the beginning of period $t = m$, if the seller is able to pay off the debt, i.e., $w_m \geq d_m$, then $\underline{q}_m \frac{(d_m - w_m)^+}{p_m} = 0$, and (2) simplifies to

$$\begin{aligned} V_m(x_m, w_m, p_m; \mathbf{d}) &= w_m - d_m + \max_{0 \leq q_m \leq \min(x_m, \ell)} p_m q_m + \rho \mathsf{E}_m U_{m+1}(x_m - q_m, P_{m+1}) \\ &= w_m - d_m + U_m(x_m, p_m). \end{aligned}$$

Thus, the statement in (i) holds in period m. Suppose it holds in period $t+1$, for some $t < m$. In period t, if $w_t \geq \sum_{i=t}^{m} \rho^{i-t} d_i$, then $\underline{q}_t = 0$ and (3) simplifies to

$$\begin{aligned} V_t(x_t, w_t, p_t; \mathbf{d}) &= \max_{0 \leq q_t \leq \min(x_t, \ell)} \rho \mathsf{E}_t V_{t+1}\big(x_t - q_t,\ \rho^{-1}(w_t + p_t q_t - d_t),\ P_{t+1}; \mathbf{d}\big) \\ &= \max_{0 \leq q_t \leq \min(x_t, \ell)} \rho \mathsf{E}_t \Big[U_{t+1}(x_t - q_t, P_{t+1}) + \rho^{-1}(w_t + p_t q_t - d_t) - \sum_{i=t+1}^{m} \rho^{i-t-1} d_i\Big] \\ &= w_t - \sum_{i=t}^{m} \rho^{i-t} d_i + \max_{0 \leq q_t \leq \min(x_t, \ell)} p_t q_t + \rho \mathsf{E}_t U_{t+1}(x_t - q_t, P_{t+1}) \\ &= w_t - \sum_{i=t}^{m} \rho^{i-t} d_i + U_t(x_t, p_t), \end{aligned}$$

where the second equality follows from the induction hypothesis because

$$\rho^{-1}(w_t + p_t q_t - d_t) \ \geq \ \rho^{-1}\Big(\sum_{i=t}^{m} \rho^{i-t} d_i - d_t\Big) = \sum_{i=t+1}^{m} \rho^{i-t-1} d_i.$$

Hence, the statement in (i) holds for all $t \leq m$.

(ii) When the asset is sold out before period $t \leq m$, the revenue-to-go is zero. If the seller is able to pay off the debt, i.e., $w_t \geq \sum_{i=t}^{m} \rho^{i-t} d_i$, then the firm's equity value is $w_t - \sum_{i=t}^{m} \rho^{i-t} d_i$. If $w_t < \sum_{i=t}^{m} \rho^{i-t} d_i$, because no asset remains, the seller will certainly go bankrupt and thus $V_t(0, w_t, p_t; \mathbf{d}) = 0$.

(iii) The monotonicity of $V_m(x_m, w_m, p_m; \mathbf{d})$ in w_m follows directly from (2); the monotonicity of $V_t(x_t, w_t, p_t; \mathbf{d})$ in w_t can be inductively proved using (3). ∎

Proof of Proposition 1: (i) We inductively prove the following two statements for all periods:

(a) It is optimal to sell the remaining asset x_t in period t if $p_t \geq R_t^o = \rho \mathsf{E}_t U_{t+1}(1, P_{t+1})$;

(b) $U_t(x_t, p_t) = U_t(1, p_t) x_t$.

In the final period T, because any unsold inventory is worthless (i.e., $U_{T+1} = 0$), it is optimal to sell everything on hand. Hence, statement (a) is true with $R_T^o = 0 = \rho U_{T+1}$. Furthermore, $U_T(x_T, p_T) = p_T x_T$, which leads to statement (b): $U_T(x_T, p_T) = U_T(1, p_T) x_T$.

Suppose statements (a) and (b) hold in period $t+1$, for some $t < T$. Then, in period t, problem

(1) becomes

$$\begin{aligned} U_t(x_t, p_t) &= \max_{0 \le q_t \le x_t} p_t q_t + \rho \mathsf{E}_t U_{t+1}(x_t - q_t, P_{t+1}) \\ &= \max_{0 \le q_t \le x_t} p_t q_t + \rho \mathsf{E}_t U_{t+1}(1, P_{t+1})(x_t - q_t) \\ &= \max_{0 \le q_t \le x_t} (p_t - R_t^o) q_t + R_t^o x_t, \end{aligned}$$

where $R_t^o = \rho \mathsf{E}_t U_{t+1}(1, P_{t+1})$. Because the above maximand is linear in q_t, the optimal decision is

$$q_t^* = \begin{cases} x_t, & \text{if } p_t \ge R_t^o, \\ 0, & \text{if } p_t < R_t^o, \end{cases}$$

which proves statement (a). Under the above q_t^*, we obtain

$$U_t(x_t, p_t) = \max\{p_t, R_t^o\} x_t. \tag{A.1}$$

Thus, we can write $U_t(x_t, p_t) = U_t(1, p_t) x_t$, as in statement (b). The induction is complete.

Because the above policy of selling all or nothing is optimal for all periods, it is optimal to sell the entire asset at once, at the time when $p_t \ge R_t^o$ occurs.

(ii) We have shown $R_T^o = 0$ and $U_t(1, p_t) = \max\{p_t, R_t^o\}$ in part (i). Thus,

$$R_{t-1}^o = \rho \mathsf{E}_{t-1} U_t(1, P_t) = \rho \mathsf{E}_{t-1} \max\{P_t, R_t^o\}.$$

Furthermore, $\rho^{t-1} R_{t-1}^o = \rho^t \mathsf{E}_{t-1} \max\{P_t, R_t^o\} \ge \rho^t \mathsf{E}_{t-1} R_t^o$.

(iii) When prices are independently distributed, $R_t^o = \rho \mathsf{E}_t U_{t+1}(1, P_{t+1})$ no longer depends on p_t and assumes a deterministic value. In such case, we can prove that $R_t^o \ge R_{t+1}^o$ by induction. This inequality holds for $t = T - 1$ since $R_T = 0$. Suppose $R_t^o \ge R_{t+1}^o$ for some $t < T$. Then, we have

$$R_{t-1}^o - R_t^o = \rho \mathsf{E} \max\{p_t, R_t^o\} - \rho \mathsf{E} \max\{P_{t+1}, R_{t+1}^o\} \ge 0,$$

where the inequality is due to the identical and independent distributions of p_t's and the induction hypothesis. This completes the induction. ∎

Proof of Proposition 2: The existence of a unique reservation price $\widehat{p}_t$ follows immediately from the statement that $R_t^o(p_t)$ increases in p_t at a rate no greater than $\rho < 1$. We inductively prove this statement.

Because $R_T^o = 0$, the statement clearly holds with $\widehat{p}_T = 0$. For $t < T$, suppose $R_{t+1}^o(p_{t+1})$ increases in p_{t+1} at a rate no greater than ρ.

Define $h(p) = \max\{p, R_{t+1}^o(p)\}$. By Proposition 1(ii), we have $R_t^o(p_t) = \rho \mathsf{E}\big[h(P_{t+1}) \mid p_t\big]$.

First, we see that $R_t^o(p_t)$ increases in p_t, because P_{t+1} stochastically increases in p_t and $h(\cdot)$ is an increasing function.

Second, we prove the upper bound on the rate of increase. Consider two arbitrary prices $p_t^a < p_t^b$.

Condition (i) in the proposition means that $P_{t+1}|p_t^a$ is stochastically smaller than $P_{t+1}|p_t^b$. By the coupling property of stochastic ordering, there exist a probability space and two random variables P^a and P^b on this space such that P^i has the same distribution as $P_{t+1}|p_t^i$, for $i = a, b$, and $P^a \le P^b$ almost surely. The induction hypothesis and the definition of $h(p)$ imply that $h(p)$ increases in p at a rate no greater than 1. Thus, we have $h(P^b) - h(P^a) \le P^b - P^a$ almost surely. Taking expectations on both sides, we have $\mathsf{E}[h(P^b)] - \mathsf{E}[h(P^a)] \le \mathsf{E}[P^b] - \mathsf{E}[P^a]$, which leads to

$$\mathsf{E}[h(P_{t+1}) \mid p_t^b] - \mathsf{E}[h(P_{t+1}) \mid p_t^a] \;\le\; \mathsf{E}[P_{t+1} \mid p_t^b] - \mathsf{E}[P_{t+1} \mid p_t^a] \;\le\; p_t^b - p_t^a,$$

where the last inequality is due to condition (ii) in the proposition. Multiplying ρ throughout, we have $R_t^o(p_t^b) - R_t^o(p_t^a) \le \rho(p_t^b - p_t^a)$. This completes the induction. ∎

Proof of Lemma 2: (i) We prove by induction. In period $m = T$, equation (2) simplifies to

$$V_T(x_T, w_T, p_T; \mathbf{d}) = (p_T x_T + w_T - d_T)^+,$$

which is jointly convex in (x_T, w_T). Suppose $V_{t+1}(x_{t+1}, w_{t+1}, P_{t+1}; \mathbf{d})$ is convex in (x_{t+1}, w_{t+1}) for some $t < T$. Then in period t, because there is no immediate debt obligation ($d_t = 0$), we have $\underline{q_t} = 0$ and (3) leads to

$$V_t(x_t, w_t, p_t; \mathbf{d}) = \max_{0 \le q_t \le x_t} \rho \mathsf{E}_t V_{t+1}\big(x_t - q_t, \rho^{-1}(w_t + p_t q_t), P_{t+1}; \mathbf{d}\big). \tag{A.2}$$

By the induction hypothesis, the objective function in (A.2) is convex in q_t. Therefore, the optimal decision q_t^* is either 0 or x_t, and we can write $V_t(x_t, w_t, p_t; \mathbf{d})$ as the maximum of two functions:

$$V_t(x_t, w_t, p_t; \mathbf{d}) = \rho \max\left\{\mathsf{E}_t V_{t+1}(x_t, \rho^{-1} w_t, P_{t+1}; \mathbf{d}),\, \mathsf{E}_t V_{t+1}(0, \rho^{-1}(w_t + p_t x_t), P_{t+1}; \mathbf{d})\right\}. \tag{A.3}$$

Because both functions inside of maximization in (A.3) are convex in (x_t, w_t), the maximum of them is also convex in (x_t, w_t). This completes the induction.

(ii) We prove by induction. The statement holds in period T because

$$V_T(x_T, w_T, p_T; \mathbf{d}) = (p_T x_T + w_T - d_T)^+ \ge p_T x_T + w_T - d_T = U_T(x_T, p_T) + w_T - d_T.$$

Suppose the statement holds in period $t+1$ for some $t < T$. Then, applying the induction hypothesis to (A.2), we have

$$\begin{aligned} V_t(x_t, w_t, p_t; \mathbf{d}) &\ge \max_{0 \le q_t \le x_t} \rho \mathsf{E}_t\big[U_{t+1}(x_t - q_t, P_{t+1}) + \rho^{-1}(w_t + p_t q_t) - \rho^{T-t-1} d_T\big] \\ &= w_t - \rho^{T-t} d_T + \max_{0 \le q_t \le x_t} p_t q_t + \rho \mathsf{E}_t U_{t+1}(x_t - q_t, P_{t+1}) \\ &= w_t - \rho^{T-t} d_T + U_t(x_t, p_t), \end{aligned}$$

which completes the induction. Lemma 1(i) proves that the equality holds when $w_t \ge \rho^{T-t} d_T$. ∎

Proof of Proposition 3: (i) The proof of Lemma 2 shows that, for any given x_t, the optimal selling quantity q_t^* is either 0 or x_t. Thus, given $x_1 = 1$, it is optimal to sell the entire asset at once, i.e., $q_t^* \in \{0, 1\}$ for $t = 1, \dots, T$. Next, we characterize the optimal selling time.

At the beginning of period T, if the asset is not yet sold ($x_T = 1$) and if $p_T \geq d_T$, then the seller should sell the asset and earn $p_T - d_T$. If $p_T < d_T$, the seller goes bankrupt regardless it sells the asset or not. Hence, $R_T = d_T$ as stated in part (i) of the proposition.

At the beginning of period $t < T$, if the asset is not yet sold ($x_t = 1$), then the seller has $w_t = 0$ and (A.3) leads to

$$\begin{aligned} V_t(1, 0, p_t; \mathbf{d}) &= \max\{\rho \mathsf{E}_t V_{t+1}(1, 0, P_{t+1}; \mathbf{d}),\ \rho \mathsf{E}_t V_{t+1}(0, \rho^{-1} p_t, P_{t+1}; \mathbf{d})\} \\ &= \max\{\rho \mathsf{E}_t V_{t+1}(1, 0, P_{t+1}; \mathbf{d}),\ \rho(\rho^{-1} p_t - \rho^{T-t-1} d_T)^+\} \\ &= \max\{\rho \mathsf{E}_t V_{t+1}(1, 0, P_{t+1}; \mathbf{d}),\ p_t - \rho^{T-t} d_T\}, \end{aligned} \tag{A.4}$$

where the second equality is due to Lemma 1(ii) and we omit $(\ )^+$ in the last equality because the first term in (A.4) $\mathsf{E}_t V_{t+1} \geq 0$. Therefore, the seller should sell the asset if $p_t \geq R_t$, where

$$R_t = \rho^{T-t} d_T + \rho \mathsf{E}_t V_{t+1}(1, 0, P_{t+1}; \mathbf{d}). \tag{A.5}$$

If we define $V_{T+1} = 0$, then (A.5) is also valid for $t = T$ because $R_T = d_T$. Following from (A.5), we have

$$\begin{aligned} R_{t-1} &= \rho^{T-t+1} d_T + \rho \mathsf{E}_{t-1} V_t(1, 0, P_t; \mathbf{d}) \\ &= \rho^{T-t+1} d_T + \rho \mathsf{E}_{t-1} \max\{\rho \mathsf{E}_t V_{t+1}(1, 0, P_{t+1}; \mathbf{d}),\ P_t - \rho^{T-t} d_T\} \\ &= \rho(\rho^{T-t} d_T) + \rho \mathsf{E}_{t-1} \max\{R_t - \rho^{T-t} d_T,\ P_t - \rho^{T-t} d_T\} \\ &= \rho \mathsf{E}_{t-1} \max\{R_t, P_t\} \end{aligned}$$

where the second equality follows from (A.4) and the third equality is due to (A.5).

If $p_t < R_t$ throughout the entire horizon, then the seller forgoes selling opportunity from periods 1 to $T - 1$ and is unable to pay off the debt in period T and thus goes bankrupt.

(ii) We can utilize the relation $R_t(p_t) = \rho \mathsf{E}_t \max\{P_{t+1}, R_{t+1}\}$ to write

$$R_t(p_t, d_T) = \rho \mathsf{E}_t \max\{P_{t+1}, \rho \mathsf{E}_{t+1} \max\{P_{t+2}, \dots\ \rho \mathsf{E}_{T-1} \max\{P_T, d_T\}\ \dots\},$$

where we emphasize the dependence of R_t on d_T. Thus, for any realized price p_t, $R_t(p_t, d_T)$ increases in d_T. Consequently, the probability of bankruptcy $\mathsf{Pr}\{P_t < R_t \text{ for all } t \in \mathcal{T}\}$ increases in d_T.

To show that τ increases in d_T almost surely, consider two debt levels with $d_T^a < d_T^b$. The previous result shows that $R_t^a \leq R_t^b$ almost surely, for $t \in \mathcal{T}$. Now compare

$$\tau^a = \inf\{t : P_t \geq R_t^a, t \in \mathcal{T}\} \qquad \text{and} \qquad \tau^b = \inf\{t : P_t \geq R_t^b, t \in \mathcal{T}\}.$$

In every period t, if $p_t \geq R_t^b$, then we must have $p_t \geq R_t^a$. In other words, p_t exceeds R_t^a no later than it exceeds R_t^b. Therefore, $\tau^a \leq \tau^b$, which proves that τ increases in d_T almost surely.

Because the debt-free results in Proposition 1 is a special case with $d_T = 0$, the above monotonicity results suggest $R_t \geq R_t^o$ and $\tau \geq \tau^o$.

(iii) The relation $R_t = \rho \mathsf{E}_t \max\{P_{t+1}, R_{t+1}\}$ in part (i) implies that $\rho^t R_t = \rho^{t+1} \mathsf{E}_t \max\{P_{t+1}, R_{t+1}\} \geq \rho^{t+1} \mathsf{E}_t R_{t+1}$. Thus, the expected discounted critical price decreases over time.

If P_t's are i.i.d., then R_t defined in (A.5) is independent of p_t and thus deterministic. Consider the difference $R_{t-1} - R_t = \rho \mathsf{E} \max\{P_t, R_t\} - \rho \mathsf{E} \max\{P_{t+1}, R_{t+1}\}$. Because P_t and P_{t+1} are i.i.d., $R_{t-1} \geq R_t$ if and only if $R_t \geq R_{t+1}$. Therefore, the series $\{R_1, R_2, \ldots, R_T\}$ is either (weakly) increasing or decreasing.

Consider an auxiliary function $f(d) = \rho \mathsf{E} \max\{d, P_t\} - d$, which strictly decreases in d because $\rho < 1$. Furthermore, $f(0) > 0$ and $f(d) < 0$ when $d \gg p_t$. By continuity and strict monotonicity of $f(d)$, there exists a unique $\overline{d}$ such that $f(\overline{d}) = 0$ or $\rho \mathsf{E} \max\{\overline{d}, P_t\} = \overline{d}$.

When $d_T = \overline{d}$, the relations $R_T = d_T$ and $R_{t-1} = \rho \mathsf{E} \max\{R_t, P_t\}$ imply that $R_t = \overline{d}$ for all $t \in \mathcal{T}$. When $d_T < \overline{d}$, we have $f(d_T) > 0$ or $\rho \mathsf{E} \max\{d_T, P_t\} > d_T$ or $R_{T-1} > R_T$. Thus, R_t decreases in t. Similarly, when $d_T > \overline{d}$, R_t increases in t. ∎

Proof of Lemma 3: When $w_t \geq \rho^{m-t} d_m$, Lemma 1(i) implies that $V_t(x_t, w_t, p_t; \mathbf{d}) = U_t(x_t, p_t) + w_t - \rho^{m-t} d_m$. Furthermore, $U_t(x_t, p_t) = U_t(1, p_t) x_t$, as shown in the proof of Proposition 1. Thus, $V_t(x_t, w_t, p_t; \mathbf{d})$ is linear in (x_t, w_t, d_m) in region $w_t \geq \rho^{m-t} d_m$.

We inductively prove the convexity of $V_t(x_t, w_t, p_t; \mathbf{d})$ in region $w_t \leq \rho^{m-t} d_m$. In period $t = m$, this region becomes $w_m \leq d_m$. To pay off the debt d_m, the seller must sell at least $\underline{q}_m = \frac{d_m - w_m}{p_m}$.

If $x_m < \underline{q}_m$, i.e., $p_m < \frac{d_m - w_m}{x_m}$, the seller goes bankrupt and $V_m(x_m, w_m, p_m; \mathbf{d}) = 0$.

If $x_m \geq \underline{q}_m$, i.e., $p_m \geq \frac{d_m - w_m}{x_m}$, equation (2) becomes

$$\begin{aligned} V_m(x_m, w_m, p_m; \mathbf{d}) &= \max_{\underline{q}_m \leq q_m \leq x_m} p_m q_m + w_m - d_m + \rho \mathsf{E}_m U_{m+1}(x_m - q_m, P_{m+1}) \\ &= \max_{\underline{q}_m \leq q_m \leq x_m} (p_m - R_m^o) q_m + R_m^o x_m + w_m - d_m \end{aligned}$$

where the last equality follows from $U_t(x_t, p_t) = U_t(1, p_t) x_t$ by the proof of Proposition 1 and $\rho \mathsf{E}_m U_{m+1}(1, P_{m+1}) = R_m^o$. The above maximization leads to $q_m^* = \underline{q}_m$ or $q_m^* = x_m$ depending on whether $p_m < R_m^o$ or not. Combing the above cases, the value function can be written as

$$V_m(x_m, w_m, p_m; \mathbf{d}) = \max\left\{p_m, R_m^o\right\} \left(x_m - \tfrac{d_m - w_m}{p_m}\right)^+, \qquad \text{for } w_m \leq d_m. \tag{A.6}$$

Thus, $V_m(x_m, w_m, p_m; \mathbf{d})$ is jointly convex in (x_m, w_m, d_m) in region $w_m \leq d_m$. See Figure 2 in the paper for an illustration.

Suppose for some $t < m$, $V_{t+1}(x_{t+1}, w_{t+1}, p_{t+1}; \mathbf{d})$ is jointly convex in (x_{t+1}, w_{t+1}, d_m) for $w_{t+1} \leq$

$\rho^{m-t-1}d_m$. We now prove the convexity of $V_t(x_t, w_t, p_t; \mathbf{d})$ in (x_t, w_t, d_m) when $w_t \le \rho^{m-t}d_m$. Because no debt payment is required before period m, equation (3) simplifies to

$$V_t(x_t, w_t, p_t; \mathbf{d}) = \max_{0 \le q_t \le x_t} \rho \mathsf{E}_t V_{t+1}\big(x_t - q_t,\ \rho^{-1}(w_t + p_t q_t),\ P_{t+1}; \mathbf{d}\big). \tag{A.7}$$

Let us define $q_t^d \overset{\text{def}}{=} (\rho^{m-t}d_m - w_t)/p_t$, the minimum selling quantity in period t that ensures debt payment in period m. Since $w_t \le \rho^{m-t}d_m$, we have $q_t^d \ge 0$. We solve the problem in (A.7) by considering two cases: $q_t^d \ge x_t$ and $q_t^d < x_t$.

If $q_t^d \ge x_t$, when q_t varies in $[0, x_t]$, we have $w_{t+1} = \rho^{-1}(w_t + p_t q_t) \le \rho^{-1}(w_t + p_t q_t^d) = \rho^{m-t-1}d_m$. Then, the induction hypothesis implies that the objective function in (A.7) is convex in q_t and, consequently, the potential optimal solutions are $q_t = 0$ and $q_t = x_t$. However, $q_t = x_t$ leads to $V_{t+1}(0,\ \rho^{-1}(w_t + p_t x_t),\ P_{t+1}; \mathbf{d}) = 0$ by Lemma 1(ii). Hence, $q_t^* = 0$ in this case.

If $q_t^d < x_t$, we divide the range $[0, x_t]$ into two intervals $[0, q_t^d]$ and $[q_t^d, x_t]$. When $q_t \in [0, q_t^d]$, using the same logic as in the case of $q_t^d \ge x_t$, the objective function in (A.7) is convex in q_t. When $q_t \in [q_t^d, x_t]$, we have $w_{t+1} \ge \rho^{m-t-1}d_m$. Then, Lemma 1 and the proof of Proposition 1 lead to $V_{t+1}(x_{t+1}, w_{t+1}, P_{t+1}; \mathbf{d}) = U_{t+1}(1, P_{t+1})x_{t+1} + w_{t+1} - \rho^{m-t-1}d_m$. Consequently, the objective function in (A.7) is linear in q_t. Therefore, the optimal solution must be among three possible values: $q_t^* \in \{0, q_t^d, x_t\}$.

- At $q_t = 0$, the objective is $\rho \mathsf{E}_t V_{t+1}(x_t,\ \rho^{-1}w_t,\ P_{t+1}; \mathbf{d})$, which is convex in (x_t, w_t, d_m) in region $w_t \le \rho^{m-t}d_m$ due to the induction hypothesis.
- At $q_t = q_t^d$, the objective is $\rho \mathsf{E}_t V_{t+1}(x_t - q_t^d,\ \rho^{m-t-1}d_m,\ P_{t+1}; \mathbf{d}) = \rho \mathsf{E}_t U_{t+1}(x_t - q_t^d,\ P_{t+1}) = R_t^o(x_t - q_t^d) = R_t^o\big(x_t - (\rho^{m-t}d_m - w_t)/p_t\big)$, which is linear in (x_t, w_t, d_m). The first equality uses Lemma 1(i) and the second equality uses $U_t(x_t, p_t) = U_t(1, p_t)x_t$ and $R_t^o = \rho \mathsf{E}_t U_{t+1}(1, P_{t+1})$.
- At $q_t = x_t$, the objective is $\rho \mathsf{E}_t V_{t+1}(0, \rho^{-1}(w_t + p_t x_t), P_{t+1}; \mathbf{d}) = (w_t + p_t x_t - \rho^{m-t}d_m)^+$, which is convex in (w_t, x_t, d_m).

Hence, for $w_t \le \rho^{m-t}d_m$, the value function can be expressed as

$$\begin{aligned} V_t(x_t, w_t, p_t; \mathbf{d}) = \max \big\{ &\rho \mathsf{E}_t V_{t+1}(x_t,\ \rho^{-1}w_t,\ P_{t+1}; \mathbf{d}),\ R_t^o\big(x_t - (\rho^{m-t}d_m - w_t)/p_t\big), \\ &w_t + p_t x_t - \rho^{m-t}d_m \big\}, \qquad \text{for } w_t \le \rho^{m-t}d_m, \end{aligned} \tag{A.8}$$

where we omit $(\ )^+$ in the last term in the maximization because the first term is non-negative. Although (A.8) is derived under $q_t^d < x_t$, it holds for $q_t^d \ge x_t$ as well, because $q_t^d \ge x_t$ implies that the first term in maximization is the largest (the other terms are non-positive), which is consistent with the result for the case of $q_t^d \ge x_t$.

Because all three terms in the maximization in (A.8) are convex in (x_t, w_t, d_m) for $w_t \le \rho^{m-t}d_m$,

$V_t(x_t, w_t, p_t; \mathbf{d})$ is convex in (x_t, w_t, d_m) for $w_t \le \rho^{m-t} d_m$, completing the induction.

To show that $V_t(x_t, w_t, p_t; \mathbf{d})$ is continuous in (x_t, w_t, d_m), we only need to show that it is continuous on the boundary $w_t = \rho^{m-t} d_m$. Substitute $w_t = \rho^{m-t} d_m$ into (A.8), we have

$$\begin{aligned} V_t(x_t, w_t, p_t; \mathbf{d}) &= \max\left\{\rho \mathsf{E}_t U_{t+1}(x_t,\, P_{t+1}),\, R_t^o x_t,\, p_t x_t\right\} \\ &= \max\left\{R_t^o x_t,\, R_t^o x_t,\, p_t x_t\right\} \\ &= \max\left\{R_t^o,\, p_t\right\} x_t \\ &= U_t(x_t, p_t), \end{aligned}$$

where the first equality follows from Lemma 1(i), the second equality uses the definition of (4), and the last equality follows from (A.1). On the other hand, for $w_t \ge \rho^{m-t} d_m$, we have shown that $V_t(x_t, w_t, p_t; \mathbf{d}) = U_t(x_t, p_t) + w_t - \rho^{m-t} d_m$. Therefore, $V_t(x_t, w_t, p_t; \mathbf{d})$ is continuous on the boundary $w_t = \rho^{m-t} d_m$, and hence it is continuous in (x_t, w_t, d_m). ∎

Proof of Proposition 4: (i) For notational convenience in this proof, we use V_t as a short notation for $V_t(x_t\!=\!1, w_t\!=\!0, P_t; \mathbf{d})$, i.e., the firm's value when no asset is sold before period t. Similarly, we write U_t for $U_t(x_t\!=\!1, P_t)$.

The value functions are derived in (A.6) and (A.8) for $t = m$ and $t < m$, respectively. If no asset is sold before period m, (A.8) also holds for $t = m$. To see this, use $V_{m+1} = 0$ defined with (5), and note that (A.8) becomes $V_m = \max\left\{0,\ R_m^o(1 - \frac{d_m}{p_m}),\ p_m - d_m\right\} = \max\left\{p_m, R_m^o\right\}\left(1 - \frac{d_m}{p_m}\right)^+$, which is consistent with (A.6). Thus, we focus on (A.8) for the rest of the proof.

When no asset is sold before period $t \le m$, (A.8) becomes

$$V_t = \max\left\{\rho \mathsf{E}_t V_{t+1},\ R_t^o(1 - \rho^{m-t} d_m/p_t),\ p_t - \rho^{m-t} d_m\right\}, \tag{A.9}$$

where $\rho \mathsf{E}_t V_{t+1}$ is the expected firm's value if selling nothing in period t, $R_t^o(1 - \rho^{m-t} d_m/p_t)$ is the expected firm's value if selling $q_t = \rho^{m-t} d_m/p_t$ to ensure debt payment, and $p_t - \rho^{m-t} d_m$ is the firm's value if selling the entire asset in period t.

If $p_t \le \rho^{m-t} d_m$, the second and third terms in (A.9) are non-positive and, therefore, $q_t^* = 0$.

If $p_t > \rho^{m-t} d_m$, all three terms in (A.9) are non-negative, and we need to compare them in pairs:

(a) $\rho \mathsf{E}_t V_{t+1} > R_t^o(1 - \rho^{m-t} d_m/p_t)$ if and only if $p_t < \frac{\rho^{m-t} d_m R_t^o}{R_t^o - \rho \mathsf{E}_t V_{t+1}}$;[4]

(b) $\rho \mathsf{E}_t V_{t+1} > p_t - \rho^{m-t} d_m$ if and only if $p_t < \rho \mathsf{E}_t V_{t+1} + \rho^{m-t} d_m$;

(c) $R_t^o(1 - \rho^{m-t} d_m/p_t) > p_t(1 - \rho^{m-t} d_m/p_t)$ if and only if $p_t < R_t^o$.

Combining (a) and (b) and the case of $p_t \le \rho^{m-t} d_m$, we see that the first term in (A.9) is the

[4]The denominator $R_t^o - \rho \mathsf{E}_t V_{t+1} > 0$ because $R_t^o = \rho \mathsf{E}_t U_{t+1}(1, P_{t+1}) = \rho \mathsf{E}_t V_{t+1}(1, 0, P_{t+1}; \mathbf{0}) > \rho \mathsf{E}_t V_{t+1}(1, 0, P_{t+1}; \mathbf{d})$, where the equality follows from Lemma 1(i) and the inequality is because any debt reduces the firm's value.

largest (thus $q_t^* = 0$) if and only if $p_t < \min\left\{\frac{\rho^{m-t} d_m R_t^o}{R_t^o - \rho \mathsf{E}_t V_{t+1}},\, \rho \mathsf{E}_t V_{t+1} + \rho^{m-t} d_m\right\}$.

Combining (b) and (c), we see that the last term in (A.9) is the largest (thus $q^* = 1$) if and only if $p_t \geq \max\{R_t^o,\, \rho \mathsf{E}_t V_{t+1} + \rho^{m-t} d_m\}$.

Let us define

$$R_t^{(1)} \stackrel{\text{def}}{=} \min\left\{\frac{\rho^{m-t} d_m R_t^o}{R_t^o - \rho \mathsf{E}_t V_{t+1}},\, \rho \mathsf{E}_t V_{t+1} + \rho^{m-t} d_m\right\}, \qquad R_t^{(2)} \stackrel{\text{def}}{=} \max\{R_t^o,\, \rho \mathsf{E}_t V_{t+1} + \rho^{m-t} d_m\}. \tag{A.10}$$

Clearly, $R_t^{(1)} \leq \rho \mathsf{E}_t V_{t+1} + \rho^{m-t} d_m \leq R_t^{(2)}$. The optimal policy is to sell $q_t^* = 0$ if $p_t < R_t^{(1)}$, sell $q_t^* = \rho^{m-t} d_m / p_t$ if $R_t^{(1)} \leq p_t < R_t^{(2)}$, and sell $q_t^* = 1$ if $p_t \geq R_t^{(2)}$.

When $q_t^* = \rho^{m-t} d_m / p_t$ occurs, the sales revenue $\rho^{m-t} d_m$ will ensure the debt payment and, therefore, the seller can maximize the expected revenue from the remaining asset. According to Proposition 1, the seller should sell the remaining asset whenever $p_t \geq R_t^o$.

(ii) By definition, $\rho \mathsf{E}_t \Delta_{t+1} = \rho \mathsf{E}_t V_{t+1} - \rho\big(\mathsf{E}_t U_{t+1} - \rho^{m-t-1} d_m\big) = \rho \mathsf{E}_t V_{t+1} - R_t^o + \rho^{m-t} d_m$. Then, $\mathsf{E}_t \Delta_{t+1} < 0$ implies the following two inequalities:

$$R_t^o > \rho \mathsf{E}_t V_{t+1} + \rho^{m-t} d_m > \frac{\rho^{m-t} d_m R_t^o}{R_t^o - \rho \mathsf{E}_t V_{t+1}}, \tag{A.11}$$

where the second inequality can be verified by expanding $(R_t^o - \rho \mathsf{E}_t V_{t+1})(\rho \mathsf{E}_t V_{t+1} + \rho^{m-t} d_m) > \rho^{m-t} d_m R_t^o$ and then canceling terms, noting that $R_t^o - \rho \mathsf{E}_t V_{t+1} > 0$ (see footnote 4). Hence, (A.10) and (A.11) together imply that $R_t^{(1)} = \frac{\rho^{m-t} d_m R_t^o}{R_t^o - \rho \mathsf{E}_t V_{t+1}} < R_t^o = R_t^{(2)}$.

Reversely, $\mathsf{E}_t \Delta_{t+1} \geq 0$ implies that $R_t^o \leq \rho \mathsf{E}_t V_{t+1} + \rho^{m-t} d_m$ and thus $R_t^{(2)} = \rho \mathsf{E}_t V_{t+1} + \rho^{m-t} d_m$. $\mathsf{E}_t \Delta_{t+1} \geq 0$ also implies that $\frac{\rho^{m-t} d_m R_t^o}{R_t^o - \rho \mathsf{E}_t V_{t+1}} \geq \rho \mathsf{E}_t V_{t+1} + \rho^{m-t} d_m$ and thus $R_t^{(1)} = \rho \mathsf{E}_t V_{t+1} + \rho^{m-t} d_m$. Therefore, $R_t^{(1)} = R_t^{(2)} \geq R_t^o$.

In particular, if $\mathsf{E}_t \Delta_{t+1} = 0$, then $R_t^o = \rho \mathsf{E}_t V_{t+1} + \rho^{m-t} d_m$. Therefore, $R_t^{(1)} = R_t^{(2)} = R_t^o$. ∎

Proof of Lemma 4: The second inequality in the lemma is immediately implied by the first, following the definition of Δ_t in (5). Below, we prove the first inequality using induction. As before, for notational convenience, we use V_t as a short notation for $V_t(x_t = 1, w_t = 0, P_t; \mathbf{d})$.

Equation (A.6) with $x_m = 1$ and $w_m = 0$ implies

$$\mathsf{E}_{m-1} V_m = \mathsf{E}_{m-1} \max\left\{p_m, R_m^o\right\}\big(1 - d_m/p_m\big)^+ = \int_{d_m}^{\infty} \max\left\{p_m, R_m^o\right\}\big(1 - d_m/p_m\big) dF(p_m | p_{m-1}).$$

Differentiating the integral with respect to d_m and letting d_m approach zero, we have

$$\begin{aligned} \lim_{d_m \to 0^+} \frac{\partial \mathsf{E}_{m-1} V_m}{\partial d_m} &= \lim_{d_m \to 0^+} \int_{d_m}^{\infty} -\frac{\max\left\{p_m, R_m^o\right\}}{p_m} dF(p_m | p_{m-1}) \\ &= \int_0^{\infty} -\frac{\max\left\{p_m, R_m^o\right\}}{p_m} dF(p_m | p_{m-1}) \;\leq\; -1. \end{aligned}$$

If $\mathsf{Pr}\{p_m < R_m^o\} > 0$, then the integrand $-R_m^o / p_m < -1$ with nonzero probability and the above inequality becomes strict.

Now suppose that for $t < m$, $\lim_{d_m \to 0^+} \frac{\partial \mathsf{E}_t V_{t+1}}{\rho^{m-t-1}\partial d_m} \le -1$. That is, there exists $\delta > 0$, such that for all $d_m \in (0, \delta)$, we have $\frac{\partial \mathsf{E}_t V_{t+1}}{\rho^{m-t-1}\partial d_m} \le -1$.

From the proof of Proposition 4, the first term in (A.9) is the largest when $p_t < R_t^{(1)}$. Hence, (A.9) leads to

$$\begin{aligned}
\mathsf{E}_{t-1} V_t &= \mathsf{E}_{t-1} \max\left\{\rho \mathsf{E}_t V_{t+1},\ R_t^o(1-\rho^{m-t}d_m/p_t),\ p_t - \rho^{m-t}d_m\right\} \\
&= \int_0^{R_t^{(1)}} \rho \mathsf{E}_t V_{t+1} dF(p_t|p_{t-1}) + \int_{R_t^{(1)}}^{\infty} \max\left\{p_t, R_t^o\right\}(1-\rho^{m-t}d_m/p_t) dF(p_t|p_{t-1}).
\end{aligned}$$

Differentiating with respect to d_m and dividing both sides by ρ^{m-t}, we have

$$\frac{\partial \mathsf{E}_{t-1} V_t}{\rho^{m-t}\partial d_m} = \int_0^{R_t^{(1)}} \frac{\partial \mathsf{E}_t V_{t+1}}{\rho^{m-t-1}\partial d_m} dF(p_t|p_{t-1}) + \int_{R_t^{(1)}}^{\infty} -\frac{\max\left\{p_t, R_t^o\right\}}{p_t} dF(p_t|p_{t-1}) + L_t \frac{\partial R_t^{(1)}}{\rho^{m-t}\partial d_m}, \tag{A.12}$$

where $L_t = \rho \mathsf{E}_t V_{t+1}(1, 0, R_t^{(1)}, \mathbf{d}) - \max\left\{R_t^{(1)}, R_t^o\right\}(1-\rho^{m-t}d_m/R_t^{(1)})$.

The last term in (A.12) actually vanishes because $L_t \equiv 0$. To see this, note that if $\mathsf{E}_t \Delta_{t+1} < 0$, the proof of Proposition 4 shows that $R_t^{(1)} = \frac{\rho^{m-t}d_m R_t^o}{R_t^o - \rho \mathsf{E}_t V_{t+1}} < R_t^o$, implying that $L_t = 0$. If $\mathsf{E}_t \Delta_{t+1} \ge 0$, the proof of Proposition 4 shows that $R_t^{(1)} = \rho \mathsf{E}_t V_{t+1} + \rho^{m-t} d_m \ge R_t^o$, which also leads to $L_t = 0$.

Applying the induction hypothesis and $-\max\left\{p_t, R_t^o\right\}/p_t \le -1$, (A.12) leads to the desired inequality:

$$\frac{\partial \mathsf{E}_{t-1} V_t}{\rho^{m-t}\partial d_m} \le \int_0^{R_t^{(1)}} (-1) dF(p_t|p_{t-1}) + \int_{R_t^{(1)}}^{\infty} (-1) dF(p_t|p_{t-1}) \ = \ -1, \qquad \forall d_m \in (0, \delta).$$

Therefore, $\lim_{d_m \to 0^+} \frac{\partial \mathsf{E}_{t-1} V_t}{\rho^{m-t}\partial d_m} \le -1$.

Finally, if $\mathsf{Pr}\{P_t < R_t^o\} > 0$ for all $t \le m$, then the first integrand in (A.12) $\frac{\partial \mathsf{E}_t V_{t+1}}{\rho^{m-t-1}\partial d_m} \le -1-\varepsilon$ for some $\varepsilon > 0$ and the second integrand $-\max\left\{p_t, R_t^o\right\}/p_t < -1$ for $p_t < R_t^o$. Thus, regardless whether $R_t^{(1)} > R_t^o$ or $R_t^{(1)} \le R_t^o$ in (A.12), we have

$$\frac{\partial \mathsf{E}_{t-1} V_t}{\rho^{m-t}\partial d_m} \le -1-\varepsilon', \qquad \forall d_m \in (0, \delta), \text{ for some } \varepsilon' > 0,$$

which leads to $\lim_{d_m \to 0^+} \frac{\partial \mathsf{E}_{t-1} V_t}{\rho^{m-t}\partial d_m} < -1$. ∎

Proof of Proposition 5: (i) In the paper, the discussion leading to Proposition 5 already proves the existence and uniqueness of the threshold debt level D_t (for $t < m$), such that $\mathsf{E}_t \Delta_{t+1} > 0$ and strictly increases in d_m if $d_m > D_t$, and $\mathsf{E}_t \Delta_{t+1} < 0$ if $d_m \in (0, D_t)$.

At $t = m$, (5) implies that $\mathsf{E}_m \Delta_{m+1} = -\mathsf{E}_m U_{m+1}(1, P_{m+1}) + \rho^{-1} d_m = \rho^{-1}(d_m - R_m^o)$. Thus, in period m, the unique threshold debt level is $D_m = R_m^o$.

(ii) For $d_m \geq D_t$, part (i) implies that $\mathsf{E}_t\Delta_{t+1} \geq 0$. Then, based on Proposition 4(ii), we have

$$R_t^{(1)} = R_t^{(2)} = \rho\mathsf{E}_t V_{t+1} + \rho^{m-t} d_m = \rho\mathsf{E}_t\Delta_{t+1} + R_t^o,$$

where the last equality follows from (4) and (5). Because $\mathsf{E}_t\Delta_{t+1}$ increases in d_m for $d_m \geq D_t$ (discussed before Proposition 5), $R_t^{(1)} = R_t^{(2)}$ increase in d_m.

For $d_m \in (0, D_t)$, we know $\mathsf{E}_t\Delta_{t+1} < 0$ from part (i), and thus $R_t^{(2)} = R_t^o$ is independent of d_m by Proposition 4(ii).

To show $R_t^{(1)} = \frac{\rho^{m-t} d_m R_t^o}{R_t^o - \rho\mathsf{E}_t V_{t+1}}$ increases in d_m, consider the denominator as a function $f(d_m) \equiv R_t^o - \rho\mathsf{E}_t V_{t+1}(1, 0, P_{t+1}, \mathbf{d})$. Note that $f(0) = 0$ and $f(d_m)$ is concave and increasing in d_m due to Lemma 2. Hence, $d_m/f(d_m)$ is increasing in d_m, which is the desired property.

Since $R_t^{(1)}$ increases in d_m, the first-selling time $\tau = \inf\{t : p_t \geq R_t^{(1)}, 1 \leq t \leq m\}$ increases in d_m. The proof is similar to Proposition 3(ii).

(iii) With independent prices, $R_t^o = \rho\mathsf{E}_t U_{t+1}$ and $\mathsf{E}_t V_{t+1}$ become deterministic. Consequently, $R_t^{(1)}$ and $R_t^{(2)}$ in (A.10) are deterministic, so is R_m^o.

Using induction, we prove that the threshold debt level D_t is constant and equal to R_m^o. We have shown $D_m = R_m^o$ in part (i). Suppose $D_t = R_m^o$ for some $t \leq m$. We next prove $D_{t-1} = R_m^o$.

When the debt is at the threshold in period t, i.e., $d_m = D_t = R_m^o$, part (i) implies that $\rho\mathsf{E}_t\Delta_{t+1} = \rho\mathsf{E}_t V_{t+1} - R_t^o + \rho^{m-t} R_m^o = 0$, which is independent of p_t. Then, using (A.9), when $d_m = R_m^o$, for any p_t, we have

$$\begin{aligned} V_t(1, 0, p_t, \mathbf{d}) &= \max\left\{\rho\mathsf{E}_t V_{t+1},\, R_t^o(1 - \rho^{m-t} R_m^o/p_t),\, p_t - \rho^{m-t} R_m^o\right\} \\ &= \max\left\{R_t^o - \rho^{m-t} R_m^o,\, R_t^o(1 - \rho^{m-t} R_m^o/p_t),\, p_t - \rho^{m-t} R_m^o\right\}. \end{aligned} \tag{A.13}$$

Note that $R_t^o - \rho^{m-t} R_m^o \geq 0$ due to Proposition 1(ii). To simplify (A.13), consider three cases:

- If $p_t \leq \rho^{m-t} R_m^o$, the second and third terms in (A.13) are non-positive and $V_t = R_t^o - \rho^{m-t} R_m^o$.
- If $\rho^{m-t} R_m^o < p_t \leq R_t^o$, then $R_t^o - \rho^{m-t} R_m^o \geq R_t^o(1 - \rho^{m-t} R_m^o/p_t) \geq p_t(1 - \rho^{m-t} R_m^o/p_t) = p_t - \rho^{m-t} R_m^o$. Thus, $V_t = R_t^o - \rho^{m-t} R_m^o$.
- If $p_t > R_t^o$, then $p_t - \rho^{m-t} R_m^o = p_t(1 - \rho^{m-t} R_m^o/p_t) > R_t^o(1 - \rho^{m-t} R_m^o/p_t) > R_t^o - \rho^{m-t} R_m^o$. Thus, $V_t = p_t - \rho^{m-t} R_m^o$.

Combining the three cases, we have $V_t = R_t^o - \rho^{m-t} R_m^o$ if $p_t \leq R_t^o$, and $V_t = p_t - \rho^{m-t} R_m^o$ if $p_t > R_t^o$. Therefore, $V_t(1, 0, p_t, \mathbf{d}) = \max\{R_t^o, p_t\} - \rho^{m-t} R_m^o$, when $d_m = R_m^o$. Taking expectation over p_t and noting that $U_t(1, p_t) = \max\{p_t, R_t^o\}$ from the proof for Proposition 1(i), we have

$$\mathsf{E}_{t-1} V_t(1, 0, P_t, \mathbf{d}) \;=\; \mathsf{E}_{t-1} U_t(1, P_t) - \rho^{m-t} R_m^o, \quad \text{ for } d_m = R_m^o.$$

Hence, $\mathsf{E}_{t-1}\Delta_t = \mathsf{E}_{t-1} V_t - (\mathsf{E}_{t-1} U_t - \rho^{m-t} d_m) = 0$ when $d_m = R_m^o$. Referring to the discussion before Proposition 5, we conclude that $D_{t-1} = R_m^o$, which completes the induction.

Because D_t is a constant and equal to R_m^o, part (i) leads to $\operatorname{sgn}(\mathsf{E}_t\Delta_{t+1}) = \operatorname{sgn}(d_m - R_m^o)$. The rest of the results follow immediately from this relation and Proposition 4(ii). ∎

Proof of Proposition 6: Define $m_0 = 0$. For $i = 1, \dots, k$, we first show that in period t with $m_{i-1} < t \le m_i$, $V_t(x_t, w_t, p_t; \mathbf{d})$ is convex in (x_t, w_t) when w_t is in each of the following intervals: $\Omega_t^i \equiv \big[0,\ \rho^{m_i - t} d_{m_i}\big]$, $\Omega_t^{i+1} \equiv \Big[\rho^{m_i-t} d_{m_i},\ \sum_{j=i}^{i+1} \rho^{m_j - t} d_{m_j}\Big]$, $\cdots$, $\Omega_t^k \equiv \Big[\sum_{j=i}^{k-1} \rho^{m_j-t} d_{m_j},\ \sum_{j=i}^{k} \rho^{m_j-t} d_{m_j}\Big]$. Note that $\Omega_t^i \cup \Omega_t^{i+1} \cup \dots \cup \Omega_t^k = \Big[0,\ \sum_{j=i}^{k} \rho^{m_j-t} d_{m_j}\Big]$. If $w_t \ge \sum_{j=i}^{k} \rho^{m_j-t} d_{m_j}$, $V_t(x_t, w_t, p_t; \mathbf{d})$ is linear in (x_t, w_t) due to Lemma 1 and $U_t(x_t, p_t) = U_t(1, p_t) x_t$ by the proof of Proposition 1.

We prove the piecewise convexity by induction. In period $t = m_k \equiv m$, the convexity of $V_m(x_m, w_m, p_m; \mathbf{d})$ in (x_m, w_m) for $w_m \in \Omega_m^k = \big[0, d_m\big]$ follows directly from Lemma 3; see (A.6).

Suppose for some t with $m_{i-1} < t < m_i$, the convexity property holds for period $t+1$, i.e., $V_{t+1}(x_{t+1}, w_{t+1}, P_{t+1}; \mathbf{d})$ is convex in (x_{t+1}, w_{t+1}) for $w_{t+1} \in \Omega_{t+1}^s = \Big[\sum_{j=i}^{s-1} \rho^{m_j-t-1} d_{m_j},\ \sum_{j=i}^{s} \rho^{m_j-t-1} d_{m_j}\Big]$, $s = i, \dots, k$. Because no payment is due in period t, (3) becomes (A.7), which is repeated here:

$$V_t(x_t, w_t, p_t; \mathbf{d}) = \max_{0 \le q_t \le x_t} \rho \mathsf{E}_t V_{t+1}\big(x_t - q_t,\ \rho^{-1}(w_t + p_t q_t),\ P_{t+1}; \mathbf{d}\big). \tag{A.14}$$

By the induction hypothesis, the above objective function is convex in q_t when $\rho^{-1}(w_t + p_t q_t) \in \Omega_{t+1}^s$, that is, when $q_t \in [0, x_t] \cap \Big[\big(\sum_{j=i}^{s-1} \rho^{m_j-t} d_{m_j} - w_t\big)/p_t,\ \big(\sum_{j=i}^{s} \rho^{m_j-t} d_{m_j} - w_t\big)/p_t\Big]$. Therefore,

$$q_t^* \in \big\{0,\ x_t\big\} \cup \bigg\{\big(\sum_{j=i}^{s} \rho^{m_j-t} d_{m_j} - w_t\big)/p_t,:\ s = i, \dots, \overline{s}\bigg\}, \tag{A.15}$$

where $\overline{s} \stackrel{\text{def}}{=} \max\Big\{s \le k : \big(\sum_{j=i}^{s} \rho^{m_j-t} d_{m_j} - w_t\big)/p_t < x_t\Big\}$.

Using (A.15), we can write (A.14) as

$$\begin{aligned} V_t(x_t, w_t, p_t; \mathbf{d}) = \max \bigg\{ & \rho \mathsf{E}_t V_{t+1}(x_t, \rho^{-1} w_t,\ P_{t+1}; \mathbf{d}),\ \Big(p_t x_t + w_t - \sum_{j=i}^{k} \rho^{m_j-t} d_{m_j}\Big)^+, \\ & \rho \mathsf{E}_t V_{t+1}\Big(x_t - \big(\sum_{j=i}^{s} \rho^{m_j-t} d_{m_j} - w_t\big)/p_t,\ \rho^{-1} \sum_{j=i}^{s} \rho^{m_j-t} d_{m_j},\ P_{t+1}; \mathbf{d}\Big), s = i, \dots, \overline{s} \bigg\}. \end{aligned} \tag{A.16}$$

To see $V_t(x_t, w_t, p_t; \mathbf{d})$ is convex in (x_t, w_t) for $w_t \in \Omega_t^s$, note that each term within the maximization of (A.16) is convex: first, $w_t \in \Omega_t^s$ implies $\rho^{-1} w_t = w_{t+1} \in \Omega_{t+1}^s$ and hence the first term is convex in (x_t, w_t) by the induction hypothesis; the second term is clearly convex in (x_t, w_t); the last term depends on (x_t, w_t) only through $x_{t+1} = x_t - \big(\sum_{j=i}^{s} \rho^{m_j-t} d_{m_j} - w_t\big)/p_t$, and thus the last term is convex in (x_t, w_t) because $V_{t+1}(x_{t+1}, w_{t+1}, P_{t+1}; \mathbf{d})$ is convex in x_{t+1} by the induction hypothesis. This completes the induction for any period t with $m_{i-1} < t < m_i$.

In the payment period $t = m_{i-1}$, if $w_t \ge d_t$, bankruptcy will not happen in period t, and the proof is identical to the case of $m_{i-1} < t < m_i$. But if $w_t \in [0, d_t)$, then q_t must satisfy $q_t \ge \underline{q}_t$ to ensure debt payment. For any $p_t < (d_t - w_t)/x_t$, the seller goes bankrupt, and the value diminishes to zero (weakly convex). For $p_t \geq (d_t - w_t)/x_t$, the proof is parallel to the case of $m_{i-1} < t < m_i$ except that the feasible region is $q_t \in [\underline{q_t}, x_t]$. This completes the induction.

To find the critical price for the first sale, consider period $t = m_1$ first. Clearly, if $p_{m_1} < d_{m_1}$, the seller will not be able to make the first payment and go bankrupt. Thus, the critical price is $R^\dagger_{m_1} = d_{m_1}$.

For $t < m_1$, the value function is given by (A.16) with $x_t = 1$, $w_t = 0$, and $i = 1$. We can express the critical prices $R^\dagger_t(p_t)$ as

$$R^\dagger_t(p_t) = p_t + \rho \mathsf{E}_t V_{t+1}(1, 0, P_{t+1}; \mathbf{d}) - \max\left\{ \left(p_t - \sum_{j=1}^{k} \rho^{m_j - t} d_{m_j}\right)^+, \right.$$
$$\left. \rho \mathsf{E}_t V_{t+1}\left(1 - \sum_{j=1}^{s} \rho^{m_j - t} d_{m_j}/p_t,\ \rho^{-1} \sum_{j=1}^{s} \rho^{m_j - t} d_{m_j},\ P_{t+1}; \mathbf{d}\right), s = 1, \ldots, \overline{s} \right\}.$$

If $p_t < R^\dagger_t(p_t)$ for all $t \leq m_1$ (i.e., $\tau = \infty$), then the seller bankrupts at the end of period m_1.

At the time of the first sale $\tau < \infty$, we have $x_\tau = 1$, $w_\tau = 0$, and (A.15) implies that the optimal selling quantity is $q^*_\tau \in \left\{ \sum_{i=1}^{j} \rho^{m_i - \tau} d_{m_i}/p_\tau : j = 1, \ldots, k \right\} \cup \{1\}$. ∎

Proof of Proposition 7: Part (i) is immediate since the seller is debt-free if $j = k$. For part (ii), the revenue from the first sale, $\sum_{i=1}^{j} \rho^{m_i - \tau} d_{m_i}$, will cover exactly the first j installments. Thus, to maximize the expected profit, the seller only needs to consider the remaining $k - j$ debt payments over a horizon of $T - \tau$ periods, constituting a new asset-selling problem. The initial wealth of this new problem is zero because the current wealth of the firm, $\sum_{i=1}^{j} \rho^{m_i - \tau} d_{m_i}$, will be used to make the first j payments that are excluded from the new problem. The initial amount of asset available for sale in this new problem is h. We can let h be the new unit of asset and scale the price to $\widetilde{P}_t = hP_t$. Then, the new problem is structurally identical to the original asset-selling problem. ∎

Proof of Proposition 8: We solve the problem defined in (1) using backward induction and prove the following property of the revenue function: For any $t \in \mathcal{T}$, $\rho \mathsf{E}_t U_{t+1}(x_{t+1}, P_{t+1})$ is piecewise linear in x_{t+1}; the slopes for the n segments $(0, \ell], (\ell, 2\ell], \ldots, ((n-2)\ell, (n-1)\ell]$ and $((n-1)\ell, 1]$ are $R^o_{t,n} \geq R^o_{t,n-1} \geq \cdots \geq R^o_{t,1}$, respectively, and the slopes satisfy part (ii) of the proposition.

The property clearly holds for $t = T$, as $U_{T+1}(.,.) \equiv 0$ and $R^o_{T,i} = 0$, $i = 1, \ldots, n$, as given in part (iii). To see the property in period $T-1$, note that $q^*_T = \min(x_T, \ell)$ and $U_T(x_T, p_T) = \min(x_T, \ell) p_T$. Hence, $\rho \mathsf{E}_{T-1} U_T(x_T, P_T)$ is piecewise linear with slope $\rho \mathsf{E}_{T-1} P_T$ for $x_T \in [0, \ell]$ and slope 0 for $x_T > \ell$. These slopes are consistent with part (iii): $R^o_{T-1,n} = R^o_{T-1}$ and $R^o_{T-1,i} = 0$ for $i < n$.

Suppose we have proved the stated property for $\rho \mathsf{E}_t U_{t+1}(x_{t+1}, P_{t+1})$. We now find the revenue function $U_t(x_t, p_t) = \max_{0 \leq q_t \leq \min(x_t, \ell)} p_t q_t + \rho \mathsf{E}_t U_{t+1}(x_t - q_t, P_{t+1})$, as defined in (1).

Consider inventory x_t in the first segment $(0,\ell]$. Solving (1) amounts to comparing p_t with $R^o_{t,n} = R^o_t$. There are two cases: (a) if $p_t \geq R^o_t$, then $q^*_t = x_t$, and the marginal value of inventory is p_t; (b) if $p_t < R^o_t$, then $q^*_t = 0$, and the marginal value of inventory is R^o_t, which is the discounted marginal value of inventory in the next period. Combining these two cases, the marginal value of inventory is $\max\{p_t, R^o_t\}$ for inventory $x_t \in (0,l]$. Thus, $\rho\mathsf{E}_{t-1}U_t(x_t,P_t)$ is linear in x_t for $x_t \in (0,\ell]$ with slope $\rho\mathsf{E}_{t-1}\max\{P_t, R^o_t\} = R^o_{t-1}$, where the equality is from Proposition 1(ii).

Next, consider inventory x_t in the i-th ($i \in \{1,\ldots,n-1\}$) segment from the *right* of the interval $(0,1]$. In solving the concave maximization problem in (1), we compare p_t with the slopes of $\rho\mathsf{E}_t U_{t+1}(x_{t+1},P_{t+1})$ given in the induction hypothesis. Since we can sell no more than ℓ per period, we need to compare p_t with only two slopes $R^o_{t,i}$ and $R^o_{t,i+1}$. The optimal selling quantity is

$$q^*_t = \begin{cases} 0, & \text{if } p_t < R^o_{t,i}; \\ x_t - \left(\left\lceil \frac{x_t}{\ell}\right\rceil - 1\right)\ell, & \text{if } R^o_{t,i} \leq p_t < R^o_{t,i+1}; \\ \ell, & \text{if } p_t \geq R^o_{t,i+1}. \end{cases} \tag{A.17}$$

For the three cases in (A.17), the marginal value of inventory is $R^o_{t,i}$, p_t, and $R^o_{t,i+1}$, respectively.[5] Combining these three cases and noting that $R^o_{t,i} \leq R^o_{t,i+1}$, the marginal value of inventory is $\text{median}\{p_t, R^o_{t,i}, R^o_{t,i+1}\}$. Thus, $\rho\mathsf{E}_{t-1}U_t(x_t,P_t)$ is linear in x_t for x_t in the i-th segment from the right, with slope $\rho\mathsf{E}_{t-1}\text{median}\{P_t, R^o_{t,i}, R^o_{t,i+1}\} = R^o_{t-1,i}$. Finally, since $R^o_{t,i}$ increases in i, $R^o_{t-1,i}$ also increases in i. Hence, the property holds for $t-1$, which completes the induction.

In the above induction, if the seller makes a sale, the selling quality is either ℓ or $x_t - \left(\left\lceil \frac{x_t}{\ell}\right\rceil - 1\right)\ell$. Hence, the asset is divided into $n = \left\lceil \frac{1}{\ell}\right\rceil$ pieces, with $n-1$ pieces of ℓ and a remainder of size $r \equiv 1-(n-1)\ell$. Regardless of when the remainder r is sold, before the i-th sale, the inventory is in the i-th segment from the right (see the list of segments at the beginning of the proof), and thus follow the optimal rule in (A.17). The i-th sale is made only when $P_t \geq R^o_{t,i}$. The selling quantities in part (i) are implied by (A.17). ∎

Proof of Lemma 5: We prove piecewise convexity by induction. In period $m = T$, the value function in (2) is $V_T(x_T, w_T, p_T; \mathbf{d}) = (p_T \min(x_T,\ell) + w_T - d_T)^+$, which is a convex function of (x_T, w_T) in two regions: $x_T \in (0,\ell]$ and $x_T \in (\ell, 1]$.

Suppose for some $t < T$, $V_{t+1}(x_{t+1}, w_{t+1}, P_{t+1}; \mathbf{d})$ is convex in (x_{t+1}, w_{t+1}) when x_{t+1} is in each of the following n regions: $(0,\ell], (\ell, 2\ell], \ldots, ((n-2)\ell, (n-1)\ell]$ and $((n-1)\ell, 1]$. Then in period t,

[5] In the last case when $p_t \geq R^o_{t,i+1}$, it is optimal to sell ℓ, and thus x_{t+1} is in the $i+1$st segment from the right. Therefore, having ε units less of inventory in period t will reduce x_{t+1} by ε and reduce the discounted expected future revenue by $R^o_{t,i+1}$.

because $d_t = 0$, the value function in (3) becomes

$$V_t(x_t, w_t, p_t; \mathbf{d}) = \max_{0 \le q_t \le \min(x_t, \ell)} \rho \mathsf{E}_t V_{t+1}(x_t - q_t, \rho^{-1}(w_t + p_t q_t), P_{t+1}; \mathbf{d}).$$

Suppose $x_t \in \big(j\ell, \min((j+1)\ell, 1)\big]$, for $j \in \{1, \ldots, n-1\}$. By the induction hypothesis, $V_{t+1}(x_t - q_t, \rho^{-1}(w_t + p_t q_t), P_{t+1}; \mathbf{d})$ is convex in q_t for $q_t \in [0, x_t - j\ell]$ and $q_t \in [x_t - j\ell, \ell]$, considered separately. Since the convex objective is maximized at the endpoints of the intervals, we have $q_t^* \in \{0, x_t - j\ell, \ell\}$. Hence, the value function can be expressed as

$$\begin{aligned} V_t(x_t, w_t, p_t; \mathbf{d}) =& \rho \max\{\mathsf{E}_t V_{t+1}(x_t, \rho^{-1} w_t, P_{t+1}; \mathbf{d}),\ \mathsf{E}_t V_{t+1}(j\ell, \rho^{-1}(w_t + (x_t - j\ell)p_t), P_{t+1}; \mathbf{d}), \\ & \mathsf{E}_t V_{t+1}(x_t - \ell, \rho^{-1}(w_t + \ell p_t), P_{t+1}; \mathbf{d})\}. \end{aligned} \tag{A.18}$$

Because all three terms in the maximization in (A.18) are convex in (x_t, w_t) for $x_t \in (j\ell, \min((j+1)\ell, 1)]$, $V_t(x_t, w_t, p_t; \mathbf{d})$ is convex in (x_t, w_t) for $x_t \in (j\ell, \min((j+1)\ell, 1)]$, for $j \in \{1, \ldots, n-1\}$.

If $x_t \in (0, \ell]$, similar logic leads to $q_t^* = \{0, x_t\}$ and the convexity of $V_t(x_t, w_t, p_t; \mathbf{d})$ for $x_t \in (0, \ell]$, which completes the induction. ∎

Proof of Proposition 9: (i) In the proof of Lemma 5, when $x_t \in (j\ell, \min((j+1)\ell, 1)]$, $j \ge 1$, the optimal decision $q_t^* \in \{0, x_t - j\ell, \ell\}$. If $q_t^* = x_t - j\ell$, then $x_{t+1} = j\ell$, and future sales quantity will be either 0 or ℓ.

The three decisions $\{0, x_t - j\ell, \ell\}$ corresponds to the three terms in the maximization in (A.18). Hence, we can express the critical prices for the first sale as

$$\begin{aligned} R_t^c(p_t) = p_t + \rho \mathsf{E}_t V_{t+1}(1, 0, P_{t+1}; \mathbf{d}) - \rho \max\big\{&\mathsf{E}_t V_{t+1}((n-1)\ell, \rho^{-1}(1 - (n-1)\ell)p_t, P_{t+1}; \mathbf{d}), \\ &\mathsf{E}_t V_{t+1}(1 - \ell, \rho^{-1}\ell p_t, P_{t+1}; \mathbf{d})\big\}. \end{aligned}$$

The first sale is triggered by $p_t \ge R_t^c(p_t)$, which proves part (i).

(ii) After the first sale $q_{\tau^c}^* \in \{\ell, r\}$ is made in period $\tau^c < T$, the problem facing the seller is to sell the remaining asset $1 - q_{\tau^c}^*$ over $T - \tau^c$ periods with a debt maturing at T. As part of the debt is covered by the first sale, the debt in the new problem is effectively $d'_T = \big(d_T - q_{\tau^c}^* p_{\tau^c} / \rho^{T - \tau^c}\big)^+$. This problem is structurally identical to the original problem. ∎